

ELLIPTIC AFFINE HECKE ALGEBRAS AND THEIR REPRESENTATIONS

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ABSTRACT. We apply equivariant elliptic cohomology to the Steinberg variety in Springer theory, and prove that the corresponding convolution algebra is isomorphic to the elliptic affine Hecke algebra constructed by Ginzburg-Kapranov-Vasserot. Under this isomorphism, we describe explicitly the cohomology classes that correspond to the elliptic Demazure-Lusztig operators. As an application, we study the Deligne-Langlands theory in the elliptic setting, and classify irreducible representations of the elliptic affine Hecke algebra. The irreducible representations are in one to one correspondence with certain nilpotent Higgs bundles on the elliptic curve. We also study representations at torsion points in type- A .

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0. INTRODUCTION

As has been observed by Ginzburg-Kapranov-Vasserot in [GKV95], generalized cohomology theories are in correspondence with 1-parameter formal groups, and the formal groups coming from 1-dimensional algebraic groups are in turn in correspondence with non-degenerate r -matrices. Motivated by the duality between representations of the quantum R -matrices and the representations of the affine Hecke algebra, it was expected that 1-parameter algebraic groups are in correspondence with affine Hecke type algebra. In [GKV97], for any 1-dimensional algebraic group, an affine Hecke-type algebra was constructed. When the algebraic group is the additive (resp. multiplicative) group, one gets the degenerate affine Hecke algebra (resp. the classical affine Hecke algebra).

In the current paper, we study the Hecke-type algebra coming from an elliptic curve, called the *elliptic affine Hecke algebra*. The main tool we use is the equivariant elliptic cohomology of the Steinberg variety. Similar to the situation of [KL87], we use cohomology theory to classify the irreducible representations of the elliptic affine Hecke algebra, and to get the character formulas. The irreducible representations are parametrized by nilpotent Higgs bundles on the elliptic curve,

Date: December 22, 2015.

2010 *Mathematics Subject Classification.* Primary 20C08; Secondary 14M15, 14F43, 55N34 .

Key words and phrases. equivariant elliptic cohomology, Steinberg variety, elliptic affine Hecke algebra.

which occurred in the work of Baranovsky-Evens-Ginzburg as a deformation of the parametrization of irreducible modules over the quantum torus [BEG03].

Without going into the details, the elliptic affine Hecke algebra is the following object. For any root datum with lattice Λ and Weyl group W , let $T = S^1 \otimes_{\mathbb{Z}} \Lambda^\vee$ be the torus, where Λ^\vee is the dual of Λ . Let E be an elliptic curve over a ring R . Then $\mathfrak{A}_{T \times S^1} := E \otimes_{\mathbb{Z}} (\Lambda^\vee \oplus \mathbb{Z})$ is an abelian variety endowed with a natural action of W . Let $\pi : \mathfrak{A}_{T \times S^1} \rightarrow \mathfrak{A}_{T \times S^1}/W$ be the quotient map. Roughly speaking, the elliptic affine Hecke algebra \mathcal{H} is a sheaf on $\mathfrak{A}_{T \times S^1}/W$, whose local sections consist of $f = \sum_{w \in W} f_w[w]$, where f_w 's are rational functions on $\mathfrak{A}_{T \times S^1}$ subject to certain conditions spelled out in Section 4.1. It is endowed with a natural structure of sheaf of algebras. Furthermore, there is a natural faithful action of \mathcal{H} on the sheaf of functions on $\mathfrak{A}_{T \times S^1}$.

0.1. Elliptic cohomology and convolution construction of the elliptic affine Hecke algebra. The axiomatic definition of the equivariant elliptic cohomology was given by Ginzburg-Kapranov-Vasserot in [GKV95]. Its construction is sketched by Grojnowski in [Gr94b], and later on established in details by many people. To give a, far from being complete, list of references that we are following, see [And00], [And03], [Ch10], [Ge06]. (See also the exposition in [Ga12].) This construction has been studied by [Lu09] in the framework of derived algebraic geometry. In particular, Lurid proved the existence of the equivariant elliptic cohomology in a much more general setting. In § 2 and § 3, we collect the basic notions and first properties of the equivariant elliptic cohomology. We present proofs for some folklore facts, including the Atiyah-Bott-Berlin-Vergne localization theorem, Atiyah-Segal completion theorem, and Quillen-Weyl-Kac pushforward formula.

For any compact Lie group G , let \mathfrak{A}_G be the moduli space of semi-stable topologically trivial principal G^{alg} bundles on the dual curve E^\vee , where G^{alg} is the corresponding split algebraic group of G over R . For example, $\mathfrak{A}_{(S^1)^n} \cong E \otimes_{\mathbb{Z}} (\mathbb{Z}^n)$, which is compatible with the notations in the previous subsection. For any connected compact Lie group G with maximal torus T and Weyl group W , we have a natural isomorphism $\mathfrak{A}_G \cong \mathfrak{A}_T/W$.

For any G -space X , the G -equivariant elliptic cohomology of X , denoted by $\mathcal{E}_G^*(X)$, is a sheaf of \mathbb{Z} -graded algebras on \mathfrak{A}_G . When $X = \text{pt}$, then $\mathcal{E}_G^i(\text{pt}) \cong \mathcal{O}_{\mathfrak{A}_G}$ for even i , and vanishes for odd i .

Let \mathcal{B} be the complete flag variety. Let \mathcal{N} be the nil-cone of the complexified Lie algebra of G , and let $\tilde{\mathcal{N}}$ be $T^*\mathcal{B}$. There is a natural map $\tilde{\mathcal{N}} \rightarrow \mathcal{N}$ which is a resolution of singularity, called the Springer resolution. There are natural actions of $G^{\mathbb{C}} \times \mathbb{C}^*$ on \mathcal{N} and $\tilde{\mathcal{N}}$, that make the Springer resolution equivariant. The fiber product $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ is called the Steinberg variety. Then $\Xi_{G \times S^1}(Z)$, certain twisted version of $\mathcal{E}_{G \times S^1}^0(Z)$, is endowed with a convolution product (defined in [GKV95]), making it a coherent sheaf of associated algebras on $\mathfrak{A}_{G \times S^1}$. (See § 3.6 for details of the definition.)

Theorem A (Theorem 5.6). There is an isomorphism $\Upsilon : \mathcal{H} \cong \Xi_{G \times S^1}(Z)$ of sheaves of algebras on $\mathfrak{A}_{G \times S^1}$, where \mathcal{H} is the elliptic affine Hecke algebra associated to the root datum of G , and $\Xi_{G \times S^1}(Z)$ the convolution algebra of the Steinberg variety.

The definition of \mathcal{H} given in [GKV97] was motivated by the calculation in equivariant K -theory. Therefore, Theorem A had been expected since then. However, no proof was known so far even in special cases. The proof we give in § 5 is similar to that in [CG97] in the K -theory case, although there are phenomena in the elliptic cohomology which do not occur in the K -theory. Moreover, in the proof of Theorem A, we found explicit cohomology classes in $\Xi_{G \times S^1}(Z)$ which correspond to the elliptic Demazure-Lusztig operators in [GKV97], under this isomorphism.

Note that there is another construction, different from but somehow parallel to the one given by [GKV97], which provides a Hecke-type algebra associated to a root datum and a 1-parameter

formal group laws (see [HMSZ12], [Zh13], and [ZZ14]). In [ZZ14], it is shown that these formal affine Hecke algebras are Borel equivariant cohomology of Steinberg varieties. In the case when the formal group law comes from an elliptic curve, the algebra constructed using the formal group law is a completion of the algebra studied in the current paper. (See also Remark 4.3.)

The study of convolution algebras of equivariant elliptic cohomology is postulated in [Gr94a]. In [GKV95], a convolution construction of the classical elliptic algebra of type-A is given. It is expected that a similar construction applied to quiver varieties will give the quantum elliptic algebra. We postpone to a subsequent paper the study of quantum elliptic algebras, and the Schur-Weyl duality between quantum elliptic algebras and elliptic affine Hecke algebras in type-A.

0.2. Representations of the elliptic affine Hecke algebra over the complex numbers.

In § 6.4, we show one application of Theorem A, i.e., a Deligne-Langlands type classification of irreducible representations of \mathcal{H} . Similar to the story of the classical affine Hecke algebra studied by Kazhdan-Lusztig [KL87] and Ginzburg [Gi85], representations of convolution algebras can be studied using the decomposition theorem. With minimal amount of notations introduced, we summarize the classification as follows.

Let E be a complex elliptic curve. For $(a, t) \in \mathfrak{A}_T \times E$, let $T(a, t) < T \times S^1$ be the smallest subgroup $T' < T \times S^1$ such that (a, t) is contained in $\mathfrak{A}_{T'} \subseteq \mathfrak{A}_T$. (See Section 4.1 for details.) We have $T(a) < T$. Let x be in the set of fixed points $\mathcal{N}^{T(a, t)}$. We define $G(a, x)$ to be the simultaneous centralizer of $T(a) < G^{\text{alg}}$ and $x \in \mathcal{N}$. Let $C(a, x)$ be the component group of $G(a, x)$. Let \mathcal{H}_t be the restriction of \mathcal{H} to the subvariety $\mathfrak{A}_T/W \times \{t\} \subseteq \mathfrak{A}_T/W \times E$.

Theorem B (Corollary 6.10.(2)). Assume $t \in E$ is a non-torsion point, then the irreducible representations of \mathcal{H}_t are in one-to-one correspondence with triples (a, x, χ) , where $a \in \mathfrak{A}_G$, $x \in \mathcal{N}^{T(a, t)}$, and χ is an irreducible representation of $C(a, x)$ which has non-zero multiplicity in $H^*(\mathcal{B}_x^{T(a)})$.

We also give a Deligne-Langlands-Lusztig type character formula in § 6.4.

Although the data parametrizing the irreducible representations of \mathcal{H} is similar to those of the affine Hecke algebra, the representations theory of the elliptic affine Hecke algebra is richer. Only in the case when $T(a, t)$ is generated by one semi-simple element, the corresponding irreducible representations of \mathcal{H} are isomorphic to those of the affine Hecke algebra of G .

In § 6.5, we show that when t is a non-torsion point, the set of triples $\{(a, x, \chi)\}$ above is in one to one correspondence with the set of $\mathcal{O}(\{0\} - \{t\})$ -valued nilpotent Higgs bundles $\text{Higgs}_t^{\text{nil}}(E^\vee)$ on E . The set $\text{Higgs}_t^{\text{nil}}(E^\vee)$ was introduced in [BEG03] (see also § 6.5). For $t = 0$, in *loc. cit.* it is shown that $\text{Higgs}_{t=0}^{\text{nil}}(E^\vee)$ parametrizes irreducible representations of the quantum torus. It was conjectured in *loc. cit.* that $\text{Higgs}_t^{\text{nil}}(E^\vee)$ parametrizes the irreducible representations of the double affine Hecke algebra of Cherednik (DAHA for short). However, in [Vas05] a classification of irreducible modules over DAHA has been achieved. There are more irreducible representations than predicted by the conjecture in [BEG03]. In a future publication of the first author and Valerio Toledano Laredo, we will show an irreducible modules over DAHA that do not occur as Higgs bundle on the elliptic curve if and only if it is *torsion*. Hence, the non-torsion irreducible integrable modules over DAHA are parametrized by Higgs bundles.

0.3. Representations in type-A. Representations of the affine Hecke algebra of type-A at roots of unity have been studied by Ariki [Ar95] and Grojnowski [Gr94a]. In particular, they proved that the Grothendieck group of affine Hecke algebra of type-A at a l -th root of unity is isomorphic to the negative half of the affine quantum group of type A_l . See [Ar95, Proposition 4.3] for details. They also proved that under this isomorphism the Lusztig canonical basis corresponds to the dual of the classes of the simple objects in the category of modules over the affine Hecke algebra at roots

of unity. This isomorphism has a categorical version, studied in [BK08], [KL09], and [R08]. More precisely, it is well-known that, based on the work of Khovanov-Lauda and Rouquier, certain category of modules over the quiver Hecke algebra categorifies the negative half of the affine quantum group of type-A. On the other hand, it is proved in [R08, Proposition 3.18] that the affine Hecke algebra at a root of unity is Morita equivalent to a suitable quiver Hecke algebra. Hence, one obtains a categorification of the Ariki-isomorphism.

Using the Khovanov-Lauda and Rouquier's work on the quiver Hecke algebra, we easily obtain the elliptic counterpart of the Ariki's theorem and its categorified version. This is carried out in § 7. We summarize the results here.

Assume $q_1, q_2 \in S^1$ are two torsion points of order n_1 and n_2 respectively, $d = \text{lcm}\{n_1, n_2\}$ and $l = n_1 \cdot n_2 / d$. Fix an isomorphism $E \cong S^1 \times S^1$ as Lie groups. Let $S_t \subset E$ be the subset consisting of $z \in E$ such that z has the form (q_1^u, q_2^v) for $u, v \in \mathbb{Z}$, and let $S_t^n \subseteq E^n$ be its product. Let \mathcal{H}_n be the elliptic affine Hecke algebra of GL_n (or equivalently of U_n). Let $\text{Mod}_t \mathcal{H}_n$ be the subcategory of finite dimensional \mathcal{H}_n -modules, whose restriction to the action of \mathcal{S} , considered as coherent sheaves on E^n , are set theoretically supported on S_t^n .

Let $\Gamma_{d,l}$ be disjoint union of l -copies of the cyclic quiver with d vertices. Let $H_n(\Gamma_{d,l})$ be the quiver Hecke algebra of $\Gamma_{d,l}$, and $\text{Mod}_0 H_n(\Gamma_{d,l})$ be certain category of modules spelled out in details in § 7.1.

Theorem C (Theorem 7.6 and Corollary 7.7). (1) There is an equivalence of abelian categories $\text{Mod}_0 H_n(\Gamma_{d,l}) \cong \text{Mod}_t \mathcal{H}_n$.

(2) This isomorphism induces an isomorphism $(U^-(\widehat{\mathfrak{sl}}_d))^{\otimes l} \cong \bigoplus_n K(\text{Mod}_t \mathcal{H}_n)^*$.

Here $\widehat{\mathfrak{sl}}_d$ is the affine Lie algebra of type A_{d-1} , and $U^-(\widehat{\mathfrak{sl}}_d)$ the negative part of its enveloping algebra. Under the isomorphism in (2), we identify explicit auto-functors on $\bigoplus_n K(\text{Mod}_t \mathcal{H}_n)$ which correspond to the Chevalley basis in $U^-(\widehat{\mathfrak{sl}}_d)^{\otimes l}$.

There is also a graded version of this theorem, in which the enveloping algebra is replaced by half of the quantum group. Under this isomorphism, the Lusztig canonical basis corresponds to the dual basis of the classes of simple objects.

Notations. We summarize the conventions of push-forwards and pull-backs we use, for the convenience of the readers.

For any map of schemes $f : X \rightarrow Y$ and quasi-coherent sheaf \mathcal{G} on Y , we use $f^* \mathcal{G}$ to denote the inverse-image of \mathcal{G} , and $f^{-1} g \in H^0(X, f^* \mathcal{G})$, the pull-back section of $g \in H^0(Y, \mathcal{G})$.

Let $p : X \rightarrow Y$ be a morphism between two topological G -spaces. By vector bundles on topological spaces, we always mean complex vector bundles, unless otherwise specified. For any G -vector bundle V on Y , its pre-image on X will be denoted by $p^* V$. Taking equivariant elliptic cohomology, we get a map between two sheaves of algebras on the moduli scheme \mathfrak{A}_G , which will be denoted by $p^\sharp : \mathcal{E}_G^*(Y) \rightarrow \mathcal{E}_G^*(X)$. The projection of the relative spectrum $\text{Spec}_{\mathfrak{A}_G} \mathcal{E}_G^0(X) \rightarrow \mathfrak{A}_G$ will be denoted by π_X^G , or simply π_X if G is clear from the context. The induced map on spectra by p^\sharp will be denoted by $p_{\mathfrak{A}} : \mathfrak{A}_G^X \rightarrow \mathfrak{A}_G^Y$. According to our convention above, the direct-image and inverse-image of quasi-coherent sheaves will be denoted by $p_{\mathfrak{A}*}$ and $p_{\mathfrak{A}}^*$. If in addition, p is proper, then there is a push-forward (Gysin map) in equivariant elliptic cohomology, $p_\sharp : \Theta(Tp) \rightarrow p_{\mathfrak{A}}^* \mathcal{O}_{\mathfrak{A}_G^Y}$ of quasi-coherent sheaves on \mathfrak{A}_G^X . By adjunction, we also have $p_{\mathfrak{A}*} \Theta(Tp) \rightarrow \mathcal{O}_{\mathfrak{A}_G^Y}$, which will also be denoted by p_\sharp .

For any compact Lie group G with maximal torus T , the natural projection $\mathfrak{A}_T \rightarrow \mathfrak{A}_G$ will be denoted by π . For any map between compact Lie groups $\phi : H \rightarrow G$, the map induced on the moduli spaces will be denoted by $\phi_{\mathfrak{A}} : \mathfrak{A}_H \rightarrow \mathfrak{A}_G$.

Acknowledgements. Most of the ideas in this paper are rooted in [GKV95]. The main theorem in this paper could in principle have appeared in *loc. cit.*. We would like to express our gratitude to Eric Vasserot for numerous times of discussions without which this paper could not have come into being. We thank Marc Levine, Valerio Toledano Laredo, and Yaping Yang for helpful discussions and correspondences. We are grateful to Eyal Markman for helps with the proof of Theorem 6.12. During the preparation of this paper, the first named author was financially supported by Fondation Sciences Mathématiques de Paris and Centre National de la Recherche Scientifique; The second named author was supported by Vladimir Chernousov and Stefan Gille, and also by PIMS. This project was conceived when both authors were visiting the Max-Planck-Institut für Mathematik.

1. THE ELLIPTIC GROUP ALGEBRA

In this section, we fix terminology and notations about line bundles on abelian varieties and theta-functions. Throughout this section, R is a commutative Noetherian ring.

1.1. The theta functions. Let E be an elliptic curve over a ring R . Let $0 : \text{Spec } R \rightarrow E$ be the zero section of the elliptic curve. Then the image of 0 is a codimension one subvariety, denoted by $\{0\}$. Let $\mathcal{O}(-\{0\})$ be the sheaf of ideas of this subvariety, hence $\mathcal{O}(-\{0\})$ is a line bundle. Its dual $\mathcal{O}(-\{0\})^\vee$ has a natural section, denoted by ϑ . Let $inv : E \rightarrow E$ be the map sending any point to its additive inverse. Then $inv^*\mathcal{O}(-\{0\}) \cong \mathcal{O}(-\{0\})$, and the natural section ϑ is sent to $-\vartheta$ under this isomorphism. In this sense, we say that ϑ is an odd function.

Example 1.1. Let E be the Tate elliptic curve over $R = \mathbb{Q}((q))$, whose equation is given by

$$y^2 + xy = x^3 + a_4(q)x + a_6(q)$$

with $a_4(q) = -5s_3(q)$ and $a_6(q) = -5s_3(q) + 7s_5(q)$, such that $s_k(q) = \sum_{n \geq 1} \frac{n^k q^n}{1 - q^n}$. Then ϑ is the Jacobi-theta function

$$\vartheta(u) = u^{\frac{1}{2}} \prod_{s > 0} (1 - q^s u) \prod_{s \geq 0} (1 - q^s u^{-1}) \frac{1}{2\pi i} \prod_{s > 0} (1 - q^s)^{-2}.$$

When E is a complex elliptic curve, i.e., \mathbb{C} quotient out by the lattice generated by 1 and some $\tau \in \mathbb{C}$, then ϑ is also the Jacobi-theta function above, evaluated at $u = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$. Here z is the coordinate of the universal cover \mathbb{C} .

Throughout this section, T will be a compact connected abelian (real) Lie group, i.e., $T \cong (S^1)^n$, where n is the *rank* of T . Let $\mathbb{X}^*(T)$ be the group of characters of T , and $\mathbb{X}_*(T)$ its dual. Let T^{alg} be the algebraic group over R which classifies maps of abelian groups from $\mathbb{X}^*(T)$ to \mathbb{G}_m . Then T^{alg} is a split algebraic torus over R , i.e., $T^{\text{alg}} \cong \mathbb{G}_m^n$. Let \mathfrak{A}_T be the R -scheme that classifies maps of abelian groups from $\mathbb{X}^*(T)$ to E , which is an abelian R -variety. Equivalently, \mathfrak{A}_T can be described as the moduli scheme of stable fiber-wise topologically trivial principal T^{alg} -bundles on the dual elliptic curve E^\vee .

We have a canonical isomorphism $\mathfrak{A}_T \cong E \otimes X_*(T)$. Any character $\lambda \in \mathbb{X}^*(T)$ induces a group homomorphism $\chi_\lambda : \mathfrak{A}_T \rightarrow E$. The subvariety $\ker \chi_\lambda$ is a divisor of \mathfrak{A}_T , whose ideal sheaf is the line bundle $\mathcal{O}(-\ker \chi_\lambda)$ on \mathfrak{A}_T . Clearly, we have $\mathcal{O}(-\ker \chi_\lambda) \cong \chi_\lambda^* \mathcal{O}(-\{0\})$. The natural section of the line bundle $\mathcal{O}(-\ker \chi_\lambda)^\vee$ is denoted by $\vartheta(\chi_\lambda)$, which, via the identification $\mathcal{O}(-\ker \chi_\lambda) \cong \chi_\lambda^* \mathcal{O}(-\{0\})$, is equal to $\chi_\lambda^{-1} \vartheta$.

Choosing coordinates $T \cong (S^1)^n$, we get a basis for $\mathbb{X}^*(T)$, say $\lambda_1, \dots, \lambda_n$, i.e., any $\lambda \in \mathbb{X}^*(T)$ can be written as $\sum_{i=1}^n n_i \lambda_i$. The coordinates $T \cong (S^1)^n$ also induces an isomorphism $\mathfrak{A}_T \cong E^n$. For any point $x \in \mathfrak{A}_T \cong E^n$, we can write $x_i \in E$ for its i -th coordinate, i.e., the image of x via the

i -th projection $E^n \rightarrow E$, $i = 1, \dots, n$. For any $\lambda = \sum_{i=1}^n n_i \lambda_i$, the morphism $\chi_\lambda : \mathfrak{A}_T \cong E^n \rightarrow E$ is given by $x = (x_1, \dots, x_n) \mapsto \sum_{i=1}^n n_i x_i \in E$. Then $\vartheta(\chi_\lambda) = \vartheta(\sum_{i=1}^n n_i x_i)$.

The map $\mathbb{X}^*(T) \rightarrow \text{Pic}(\mathfrak{A}_T)$, sending λ to $\mathcal{O}(-\ker \chi_\lambda)$, is not a group homomorphism. Instead, we have the following.

Lemma 1.2. (1) *For any characters λ_1 and λ_2 of T , the following diagram commutes*

$$\begin{array}{ccc} \mathfrak{A}_T & \xrightarrow{\chi_{\lambda_1} \times \chi_{\lambda_2}} & E \times E \\ & \searrow \chi_{\lambda_1 \otimes \lambda_2} & \downarrow \mu \\ & & E \end{array}$$

where $\mu : E \times E \rightarrow E$ is the addition of E . Let AD be the subvariety of $E \times E$ consisting of $(x, -x)$ with $x \in E$, then $\mathcal{O}(-\ker \chi_{\lambda_1 \otimes \lambda_2}) \cong (\chi_{\lambda_1} \times \chi_{\lambda_2})^* \mathcal{O}_{E \times E}(-AD)$.

(2) *The line bundle $\mathcal{O}(-\ker \chi_{\lambda^\vee})$ is canonically isomorphic to $\mathcal{O}(-\ker \chi_\lambda)$. Via this isomorphism, $\vartheta_{\chi_{\lambda^\vee}} = -\vartheta_{\chi_\lambda}$.*

Let $\text{Rep}(T)$ be the free abelian group of virtual T -characters, i.e., $\text{Rep}(T) \cong \mathbb{Z}[\mathbb{X}^*(T)]$. It follows from Lemma 1.2.(1) that the set map $\chi : \mathbb{X}^*(T) \rightarrow \text{Pic}(\mathfrak{A}_T)$ induces a homomorphism of abelian groups $\chi : \mathbb{Z}[\mathbb{X}^*(T)] \rightarrow \text{Pic}(\mathfrak{A}_T)$, where $\mathbb{Z}[\mathbb{X}^*(T)]$ is considered as a free abelian group with the additive structure.

Definition 1.3. For any $\lambda \in \mathbb{Z}[\mathbb{X}^*(T)]$, we denote $\mathcal{O}(-\ker \chi(\lambda))$ by \mathcal{L}_λ . Let $\tilde{\mathcal{S}} := \bigoplus_{\lambda \in \mathbb{Z}[\mathbb{X}^*(T)]} \mathcal{L}_\lambda$. Note that the group structure of $\text{Pic}(\mathfrak{A}_T)$ makes $\tilde{\mathcal{S}}$ into a sheaf of algebras on \mathfrak{A}_T .

The theta function $\vartheta(\chi_\lambda)$ is a section of \mathcal{L}_λ^\vee .

1.2. Looijenga's ring of theta-functions. For any irreducible reduced root system R in a real For any integer n , let $E^{(n)}$ be the n -th symmetric product E^n/\mathfrak{S}_n . The following lemma is clear.

Lemma 1.4. *Let $E^{(n-1)} \times E \rightarrow E^{(n)} \times E$ be the map given by the product of the symmetrization $E^{(n-1)} \times E \rightarrow E^{(n)}$ and the projection to the second factor $E^{(n-1)} \times E \rightarrow E$. The image of this map is a divisor in $E^{(n)} \times E$, denoted by D . Let $\iota : E^{(n)} \rightarrow E^{(n)} \times E$ be the embedding into the first factor. Then*

$$\iota^* \mathcal{O}(-D) \cong \mathcal{O}(-C),$$

where $C \subseteq E^{(n)}$ is the divisor given by $\{\{x_1, \dots, x_r\} \in E^{(n)} \mid x_i = 0 \text{ for some } i\}$.

Let $p_i : E^n \rightarrow E$ be the projection onto the i -th factor, and $\pi : E^n \rightarrow E^{(n)}$ be the symmetrization map. Note that the natural action of \mathfrak{S}_n on $\otimes_{i=1}^n p_i^* \mathcal{O}(-\{0\})$ fixes the fiber of $0 \in E^n$. Hence, by [Lo77], there is a natural isomorphism of line bundles $(\pi_*(\otimes_{i=1}^n p_i^* \mathcal{O}(-\{0\})))^{\mathfrak{S}_n} \cong \mathcal{O}(-C)$ on $E^{(n)}$. The section $\prod_{i=1}^n \vartheta(x_i)$ of $\otimes_{i=1}^n p_i^* \mathcal{O}(-\{0\})^\vee$ induces a natural section of $\mathcal{O}(-C)^\vee$, denoted by ϑ^{U_n} .

Let $\rho : T \rightarrow U_r$ be a representation of T . One has r characters of T , denoted by $\lambda_1, \dots, \lambda_r$. Let $\chi_\rho : E^n \rightarrow E^r$ be the product of χ_{λ_i} . Its composition with $E^r \rightarrow E^{(r)}$ is still denoted by χ_ρ . Define \mathcal{L}_ρ to be the pull-back of the line bundle $\mathcal{O}(-C)$ on $E^{(r)}$ via the map χ_ρ , and let $\vartheta(\chi_\rho)$ be the section of \mathcal{L}_ρ^\vee on E^n which is the pull-back via χ_ρ of the section ϑ^{U_r} of $\mathcal{O}(-C)^\vee$ on $E^{(r)}$.

The following lemma is an easy consequence of Lemma 1.2. See also [GKV95, § 1.8].

Lemma 1.5. *Let $\rho_1 : T \rightarrow U_{r_1}$ and $\rho_2 : T \rightarrow U_{r_2}$ be two representations of T . Let $\oplus : E^{(r_1)} \times E^{(r_2)} \rightarrow E^{(r_1+r_2)}$ be the symmetrization map, and let $\otimes : E^{(r_1)} \times E^{(r_2)} \rightarrow E^{(r_1 r_2)}$ be the map given*

by $(\{x_1, \dots, x_{r_1}\}, \{y_1, \dots, y_{r_2}\}) \mapsto \{x_i + y_j\}$. Then the following diagrams commute

$$\begin{array}{ccc} E^n & \xrightarrow{\chi_{\rho_1} \times \chi_{\rho_2}} & E^{(r_1)} \times E^{(r_2)} \\ & \searrow \chi_{\rho_1 \oplus \rho_2} & \downarrow \oplus \\ & & E^{(r_1+r_2)} \end{array}$$

and

$$\begin{array}{ccc} E^n & \xrightarrow{\chi_{\rho_1} \times \chi_{\rho_2}} & E^{(r_1)} \times E^{(r_2)} \\ & \searrow \chi_{\rho_1 \otimes \rho_2} & \downarrow \otimes \\ & & E^{(r_1 r_2)}. \end{array}$$

Moreover, $\vartheta(\chi_{\rho_1 \oplus \rho_2}) = \vartheta(\chi_{\rho_1}) \otimes \vartheta(\chi_{\rho_2})$ as sections of $\mathcal{L}_{\rho_1 \oplus \rho_2}^\vee \cong \mathcal{L}_{\rho_1}^\vee \otimes \mathcal{L}_{\rho_2}^\vee$.

The following is a direct consequence of Definition 1.3 and Lemma 1.5.

Corollary 1.6. *For any T -representation $\rho : T \rightarrow U_r$, the theta function $\vartheta(\chi_\rho)$ is a section of the sheaf $\tilde{\mathcal{S}}$ on \mathfrak{A}' .*

Proof. For each character λ of T , we know $\vartheta(\chi_\lambda)$ is a section of a direct summand of $\tilde{\mathcal{S}}$. Any representation $\rho : T \rightarrow U_r$ decomposes into direct sum of characters of T . By Lemma 1.5, $\vartheta(\chi_\rho)$ is the tensor of the theta-functions associated to these characters. Hence, $\vartheta(\chi_\rho)$ is a section of $\tilde{\mathcal{S}}$. \square

2. LURIE'S APPROACH TO T -EQUIVARIANT COHOMOLOGY

This section is a very brief reminder of Lurie's survey [Lu09], adjusted to our purpose. We recall Lurie's existence theorem. We prove that the Atiyah-Bott-Berlin-Vergne localization theorem and the Atiyah-Segal completion theorem are formal consequences of Lurie's theorem. These are folklore facts which are not in the literature. We also spell out the definition of Chern character in Lurie's theory, a version of which will be used later in this paper.

2.1. Derived schemes and equivariant cohomology theory. Recall that any E_∞ -ring spectrum A defines a cohomology theory, i.e., a contravariant functor from the category of topological spaces to the category of graded commutative rings, sending any X to $\bigoplus_{i \in \mathbb{Z}} \text{Hom}(X, \Sigma_{S^1}^i A)$. Denote $A^i(X) = \text{Hom}(X, \Sigma_{S^1}^i A)$.

Definition 2.1. An E_∞ -ring spectrum A is said to be even if $A^{2i-1}(\text{pt}) = 0$ for any integer i , and we say A is periodic if it is endowed with an invertible element $p \in \pi_2 A$.

If A is periodic, then for any topological space X , multiplication by $p \in \pi_2 A$ induces isomorphisms $A^i(X) \cong A^{i+2}(X)$ for all i .

Let A be an E_∞ -ring spectrum. Then $\pi_0 A$ is a commutative ring and $\pi_n(A)$ is a $\pi_0 A$ -module. The Zariski spectrum $\text{Spec } A$ of A is a pair $(\text{Spec}(\pi_0 A), \mathcal{O}_{\text{Spec } A})$, where $\text{Spec}(\pi_0 A)$ is the classical scheme, and $\mathcal{O}_{\text{Spec } A}$ is certain structure sheaf of E_∞ -ring spectra. A derived scheme X is a topological space endowed with a sheaf of E_∞ -ring spectra \mathcal{O}_X , which is locally isomorphic to the Zariski spectrum $\text{Spec } A$ of some E_∞ -ring spectrum A (see [Lu09, § 2.2]). The underlying classical scheme of X is denoted by X_0 . A morphism of derived schemes $f : X \rightarrow Y$ is a morphism of schemes $X_0 \rightarrow Y_0$ together with a morphism of sheaves of E_∞ -ring spectra $\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$. Most of the notions from scheme theory are defined in the derived setting as well, such as fiber product, flatness of morphisms, quasi-coherent sheaves and coherent sheaves.

If there is a morphism of derived schemes $f : X \rightarrow Y$, we say that X is over Y , or X is a Y -scheme. A commutative A -group is a flat A -scheme \mathbb{G} endowed with the structure of a group-object in the category of A -schemes. Its underlying scheme is denoted by \mathbb{G}_0 . Let Ω_E be the sheaf of relative Kähler differentials on \mathbb{G}_0 along the structure map $\mathbb{G}_0 \rightarrow \mathrm{Spec}(\pi_0 A)$, and let ω be the pull-back $z^* \Omega_E$ along the identity section $z : \mathrm{Spec}(\pi_0 A) \rightarrow \mathbb{G}_0$. For any A -scheme S , denote $\mathbb{G}(S) = \mathrm{Hom}_A(S, \mathbb{G})$.

Definition 2.2. A pre-orientation of a commutative A -group \mathbb{G} is an element in $\pi_2(\mathbb{G}(A))$. It induces a morphism of $\pi_0 A$ -modules $\beta : \omega \rightarrow \pi_2 A$. We say a pre-orientation is an orientation if \mathbb{G}_0 is smooth of relative dimension one over $\mathrm{Spec} \pi_0 A$, and β induces isomorphisms $\pi_n A \otimes_{\pi_0 A} \omega \rightarrow \pi_{n+2} A$ for any n .

Let T be a torus. Let $\mathfrak{A}_T^{\mathrm{der}}$ be the derived A -scheme which classifies maps of abelian groups from $\mathbb{X}^*(T)$ to \mathbb{G} . We will denote its underlying classical scheme by \mathfrak{A}_T . For example, when $T = (S^1)^n$, then $\mathfrak{A}_T^{\mathrm{der}} \cong \mathbb{G}^n$. We say a topological T -space X is finite if the following property is satisfied: there is a stratification

$$\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$$

with $X_{i+1} = X_i \amalg_{(T/T_i) \times S^{k-1}} ((T/T_i) \times D^k)$, where T_i is some closed subgroup of T , S^{k-1} is the $(k-1)$ -sphere, and D^k is the corresponding k -disk.

Theorem 2.3. [Lu09, Theorem 3.2] *For each compact abelian Lie group T , there exists a contravariant functor \mathcal{F}_T from the category of finite T -spaces to the category of quasi-coherent sheaves on $\mathfrak{A}_T^{\mathrm{der}}$, satisfying the following properties:*

- (1) *It maps T -equivariant homotopy equivalences to equivalences of quasi-coherent sheaves;*
- (2) *For any fixed T , the functor \mathcal{F}_T maps finite homotopy colimits of T -spaces to homotopy limits of quasi-coherent sheaves;*
- (3) $\mathcal{F}_T(pt) = \mathcal{O}_{\mathfrak{A}_T^{\mathrm{der}}}$;
- (4) *For any closed subgroup $T \subseteq T'$, and T -space X , define $X' = (X \times T')/T$ with the induced T' -action. Then $\mathcal{F}_{T'}(X') = \phi_{\mathfrak{A}*} \mathcal{F}_T(X)$ where $\phi_{\mathfrak{A}}$ is the natural embedding $\mathfrak{A}_T^{\mathrm{der}} \rightarrow \mathfrak{A}_{T'}^{\mathrm{der}}$ induced by the inclusion $\phi : T \rightarrow T'$.*

One observes that $\mathcal{F}_T(X)$ is automatically a sheaf of E_∞ -ring spectra on $\mathfrak{A}_T^{\mathrm{der}}$, see [Lu09, § 3.4]. For any finite T -space X , we define $\mathcal{F}_T^*(X) = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}_T^i(X)$, a sheaf of graded commutative algebras on the scheme \mathfrak{A}_T with $\mathcal{F}_T^i(X) = \pi_i(\mathcal{F}_T(X))$. In particular, the non-equivariant theory $\mathcal{F}_{\{1\}}$ is an oriented cohomology theory in the classical sense.

Let X be a finite T -space, and let V be a T -vector bundle on X . Let $Th(V)$ be the Thom space, i.e., the quotient of the disk bundle $D(V)$ by the sphere bundle $S(V)$. The following is the Thom isomorphism Theorem.

Proposition 2.4. [Lu09, Proposition 3.2] *Let V be a finite dimensional unitary representation of T . Then*

- (1) *the quasi-coherent sheaf $\mathcal{F}_T(Th(V))$ is a line bundle on $\mathfrak{A}_T^{\mathrm{der}}$;*
- (2) *for any finite T -space X , the natural map $\mathcal{F}_T(Th(V)) \otimes \mathcal{F}_T(X) \rightarrow \mathcal{F}_T(Th(V) \times X)$ is an equivalence.*

Example 2.5. [Lu09, § 3.4] For simplicity, assume A is even and periodic. Let $T = S^1$, and let V be the 1-dimensional representation on which T acts by scaling. In this case, we have $\mathfrak{A}_T^{\mathrm{der}} \cong \mathbb{G}$. Also $\mathcal{F}_T(Th(V))$ is the homotopy fiber of the map $\mathcal{F}_T(DV) \rightarrow \mathcal{F}_T(SV)$. Since DV is contractible,

$\mathcal{F}_T(Th(V))$ is the homotopy fiber of the following map of quasi-coherent sheaves on \mathfrak{A}_T^{der} :

$$\mathcal{O}_{\mathfrak{A}_T^{der}} \rightarrow \mathcal{F}_T(SV),$$

with $\mathcal{F}_T(SV)$ homotopy equivalent to the structure sheaf of the identity section of \mathfrak{A}_T^{der} . Therefore, we can identify $\mathcal{F}_T(Th(V))$ with the ideal sheaf of the zero section of \mathbb{G} . Via this identification, the map $\mathcal{F}_T(Th(V)) \rightarrow \mathcal{O}_{\mathfrak{A}_T^{der}}$ corresponds to the natural section of the line bundle $\mathcal{F}_T(Th(V))^\vee$ that vanishes of order one at the identity section of \mathfrak{A}_T^{der} .

In the rest of § 2, we deduce some formal consequences of Theorem 2.3.

2.2. Localization. First, we show that it follows from Lurie's results that the T -equivariant cohomology theory satisfies the Atiyah-Bott-Berline-Vergne localization theorem, and consequently the stalks of the sheaf \mathcal{F}_T can be identified. The proof of the localization theorem in [BV97] for singular cohomology goes almost non-changed in this setting. Nevertheless, for completeness we include the proof here.

For any closed subgroup $T' < T$, we identify $\mathfrak{A}_{T'}$ with its image in \mathfrak{A}_T . For any non-trivial character $\chi : T \rightarrow S^1$, let T_χ be the kernel of χ . We call the image of \mathfrak{A}_{T_χ} in \mathfrak{A}_T the weight hyperplane corresponding to χ .

Lemma 2.6. *Let X be a finite T -space and $\Gamma < T$ be a closed subgroup. Then the induced pull-back $i_\Gamma^\# : \mathcal{F}_T(X) \rightarrow \mathcal{F}_T(X^\Gamma)$ is an isomorphism outside a finite union of weight hyperplanes.*

Proof. We start with the case when $X^\Gamma = \emptyset$. We would like to show that $\mathcal{F}_T(X)$ is supported on a finite union of weight hyperplanes. By our assumption on X , we can write X as a finite union $\cup_{i=1}^n X_i$ with each X_i admitting a T -equivariant map $X_i \rightarrow T/\Gamma_i$ for some closed subgroup $\Gamma_i < T$ that does not contain Γ . By Theorem 2.3.(4), $\mathcal{F}_T(T/\Gamma_i)$ and hence $\mathcal{F}_T(X_i)$ itself is supported on the image of $\mathfrak{A}_{\Gamma_i} \subseteq \mathfrak{A}_T$, which is contained in some weight hyperplane (by lifting any non-trivial character of T/Γ_i to T). Therefore, $\mathcal{F}_T(X)$ is supported on the union of these weight hyperplanes, according to Theorem 2.3.(2).

In general, let $U \supset X^\Gamma$ be a T -invariant tubular neighbourhood of X^Γ in X , and let $Z = X \setminus U$ so that Z is a T -stable closed subset without Γ -fixed points. The support of $\mathcal{F}_T(Z)$ is contained in a finite union of weight hyperplanes, and outside this union, the natural restriction $\mathcal{F}_T(X) \rightarrow \mathcal{F}_T(U)$ is an isomorphism. \square

For any $a \in \mathfrak{A}_T$, let

$$(1) \quad T(a) := \cap_{a \in \mathfrak{A}_{T'}} T', \quad i_a : X^{T(a)} \rightarrow X.$$

Theorem 2.7. *Taking the stalks at the point a , the map $i_a^\#$ induces an isomorphism $\mathcal{F}_T(X)_a \rightarrow \mathcal{F}_T(X^{T(a)})_a$.*

Proof. It suffices to show that a does not belong to those weight hyperplanes constructed in the proof of Lemma 2.6. However, for a to be contained in the hyperplane $\mathfrak{A}_{\Gamma_i} \subseteq \mathfrak{A}_T$, the subgroup Γ_i has to contain $T(a)$. This contradicts with the fact that the $T(a)$ -action on $X \setminus X^{T(a)}$ is fixed-point free. \square

As in [Ga12, § 2.2], we can calculate the stalk at every point. Let $a \in \mathfrak{A}_T$ be an A -point, i.e., it defines a map $a : \mathbb{X}^*(T) \rightarrow \mathbb{G}(A)$. For any derived A -scheme S , the map a extends to a map $a : \mathbb{X}^*(T) \rightarrow \mathbb{G}(S)$. Since \mathbb{G} is an A -group, the set $\text{Hom}(\mathbb{X}^*(T), \mathbb{G}(S))$ is a (topological) abelian group. In particular, translation by a induces an automorphism t_a of $\text{Hom}(\mathbb{X}^*(T), \mathbb{G}(S))$, which in turn amounts to an automorphism t_a of \mathfrak{A}_T^{der} , still called the translation by a . For any finite T -space X , the translation t_a induces a map on stalks $\mathcal{F}_T(X)_a \rightarrow \mathcal{F}_T(X)_0$.

Corollary 2.8. *For any $a \in \mathfrak{A}_T$, we have $\mathcal{F}_T(X)_a \cong \mathcal{F}_T(X^{T(a)})_0$.*

Proof. It suffices to show that the map

$$\mathcal{F}_T(X^{T(a)})_a \rightarrow \mathcal{F}_T(X^{T(a)})_0$$

induced by t_a is an isomorphism. We have $\mathcal{F}_T(X^{T(a)}) \cong \phi_{\mathfrak{A}}^* \mathcal{F}_{T/T(a)}(X^{T(a)})$, where $\phi_{\mathfrak{A}} : \mathfrak{A}_T^{\text{der}} \rightarrow \mathfrak{A}_{T/T(a)}^{\text{der}}$ is induced by the quotient $\phi : T \rightarrow T/T(a)$. Naturally $\phi_{\mathfrak{A}} = \phi_{\mathfrak{A}} \circ t_a$, hence the statement follows. \square

2.3. Completion. Let $\{E_T^n \mid n \in \mathbb{N}\}$ be the Borel construction of classifying spaces, that is, a system of finite T -spaces such that the T -actions are free, and each E_T^n is contractible in E_T^N for N big enough. The following theorem is a generalization of the Atiyah-Segal completion theorem. It is true for any periodic ring spectrum A . For simplicity, we assume further that A is even and periodic.

Proposition 2.9. *Let A be even and periodic. Assume $\mathcal{F}_T(X)$ is a coherent sheaf on $\mathfrak{A}_T^{\text{der}}$. Let I be the sheaf of ideals on \mathfrak{A}_T corresponding to the identity point $1 \in \mathfrak{A}_T$. The natural map $\mathcal{F}_T^*(X) \rightarrow \mathcal{F}_T^*(X \times_T E_T^n)$ induces an isomorphism*

$$\mathcal{F}_T^*(X)_I^\wedge \cong \varprojlim \mathcal{F}_T^*(X \times_T E_T^n),$$

where the left hand side is the completion with respect to the I -adic topology.

Proof. We follow the original proof in [AS69, § 3]. First we prove this in the special case when $T = S^1$. Let $E_T^n = SV^n$ where V^n is a complex vector space of dimension n with S^1 -action by scaling, and SV^n is the unit sphere in V^n . Let DV^n be the unit disk. The Thom space $Th(V^n)$ is the homotopy cofiber of the inclusion $SV^n \rightarrow DV^n$. Then there is a long exact sequence of coherent sheaves on \mathfrak{A}_T :

$$\rightarrow \mathcal{F}_T^i(X \times_T Th(V^n)) \rightarrow \mathcal{F}_T^i(X \times_T DV^n) \rightarrow \mathcal{F}_T^i(X \times_T SV^n) \rightarrow \mathcal{F}_T^{i+1}(X \times_T Th(V^n)) \rightarrow .$$

By Proposition 2.4 and Example 2.5, the second map in the sequence above can be identified with $\mathcal{F}_T^i(X) \otimes_{\mathfrak{A}_T} \mathcal{L}^n \rightarrow \mathcal{F}_T^i(X)$ given by the natural section ϑ^{-n} of the line bundle \mathcal{L}^{-n} on \mathfrak{A}_T . Note that the image of ϑ^{-1} in $\mathcal{O}_{\mathfrak{A}_T}$ defines the ideal sheaf I , so the cokernel of $\vartheta^{-n} : \mathcal{F}_T^i(X) \otimes \mathcal{L}^n \rightarrow \mathcal{F}_T^i(X)$ is isomorphic to $\mathcal{F}_T^i(X)/(I^n)$. Denote the kernel of ϑ^{-n} by $\mathcal{F}_n^i \subset \mathcal{F}_T^i(X \times_T Th(V^n))$. We then have a short exact sequence

$$0 \rightarrow \mathcal{F}_T^i(X)/(I^n) \rightarrow \mathcal{F}_T^i(X \times_T SV^n) \rightarrow \mathcal{F}_n^{i+1} \rightarrow 0.$$

To prove the lemma, we need to show the existence of β_k in the following diagram for some k

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}_T^i(X)/(I^{n+k}) & \longrightarrow & \mathcal{F}_T^i(X \times_T SV^{n+k}) & \longrightarrow & \mathcal{F}_{n+k}^{i+1} \longrightarrow 0 \\ & & \downarrow & \swarrow \beta_k & \downarrow & & \downarrow \vartheta^{-k} \\ 0 & \longrightarrow & \mathcal{F}_T^i(X)/(I^n) & \longrightarrow & \mathcal{F}_T^i(X \times_T SV^n) & \longrightarrow & \mathcal{F}_n^{i+1} \longrightarrow 0. \end{array}$$

Since $\mathcal{F}_T^i(X)$ is coherent, \mathcal{F}_n^i stabilizes for n large enough. So ϑ^{-k} annihilates \mathcal{F}_{n+k}^i for n large enough, and β_k exists.

By the induction argument in [AS69, § 3, Step 2], the completion theorem for general torus T can be reduced to the case when $T = S^1$. This finishes the proof. \square

2.4. The Chern character. Recall that an orientation on the A -group \mathbb{G} makes the cohomology theory defined by the spectrum A into an oriented cohomology theory in the classical sense. Equivalently, it implies that for any proper map $f : X \rightarrow Y$, there is a push-forward

$$f_A : A^*(X) \rightarrow A^*(Y)$$

which is a morphism of $A^*(Y)$ -modules. Moreover, there is an associated formal group law $F(u, v) \in A^*(\text{pt})[[u, v]]$ determined by

$$c_1^A(\mathcal{L}_1 \otimes \mathcal{L}_2) = F(c_1^A(\mathcal{L}_1), c_1^A(\mathcal{L}_2)), \text{ where } \mathcal{L}_1, \mathcal{L}_2 \text{ are line bundles over } X,$$

where c_1^A is the first Chern class in the cohomology theory A . Assuming $A^*(\text{pt})$ is a \mathbb{Q} -algebra, the formal group law associated to A has an exponential $\mathfrak{l}_A(t) \in A^*(\text{pt})[[t]]$, that is characterized by $F(u, v) = \mathfrak{l}_A(\mathfrak{l}_A^{-1}(u) + \mathfrak{l}_A^{-1}(v))$. This exponential induces a natural isomorphism between A^* and H^* as functors to the category of graded commutative rings, denoted by ch^A , called the non-equivariant Chern character. It is determined by the following property: for any compact smooth manifold X , the ring homomorphism $\text{ch}^A : A^*(X) \rightarrow H^*(X; A^*(\text{pt}))$ sends $c_1^A(\mathcal{L})$ to $\mathfrak{l}_A(c_1^H(\mathcal{L}))$ for any line bundle \mathcal{L} over X .

Moreover, there is a Riemann-Roch type theorem. For any line bundle \mathcal{L} on a smooth manifold X , define $Td_A(\mathcal{L}) = \frac{c_1^H(\mathcal{L})}{\mathfrak{l}_A(c_1^H(\mathcal{L}))} \in H^*(X; A^*(\text{pt}))$, or more generally, for any rank- n vector bundle V on X with Chern roots $\{x_1, \dots, x_n\}$, define $Td_A(V) = \prod_i \frac{x_i}{\mathfrak{l}_A(x_i)}$.

Theorem 2.10. *For any proper map $f : X \rightarrow Y$ between smooth manifolds, let Tf be the relative tangent bundle. We have, for any $\alpha \in A^*(X)$,*

$$\text{ch}^A(f_A(\alpha)) = f_H(\alpha \cdot Td_A(Tf)).$$

For any $a \in \mathfrak{A}_T$, we define the localized Chern character at a , denoted by ch_a^A , to be the following composition

$$\mathcal{F}_T^*(X)_a \xrightarrow{i^*} \mathcal{F}_T^*(X^{T(a)})_a \xrightarrow{i_a^*} \mathcal{F}_T^*(X^{T(a)})_0 \rightarrow \lim_{\leftarrow} \mathcal{F}^*(X^{T(a)} \times_T E_T^n)_0 \xrightarrow[\text{ch}^A]{\cong} \lim_{\leftarrow} H^*(X^{T(a)} \times_T E_T^n; A^*(\text{pt}))_0$$

where the third map induces an isomorphism on completion by Proposition 2.9.

Summarizing the discussions above, we have the following

Corollary 2.11. *Assume that $A^*(\text{pt})$ is a \mathbb{Q} -algebra, and let $\text{ch}_a^A : \mathcal{F}_T^*(X)_a \rightarrow H_T^*(X^{T(a)}; A^*(\text{pt}))$ be the localized Chern character as above. Then it induces an isomorphism on the completion $\mathcal{F}_T^*(X)_a^\wedge \cong H_T^*(X^{T(a)}; A^*(\text{pt}))_0^\wedge$.*

3. EQUIVARIANT ELLIPTIC COHOMOLOGY THEORY

In this section, we collect some basic notions, constructions and properties of equivariant elliptic cohomology from [Lu09] and [GKV95]. We do not claim anything original in this section. Nevertheless, in § 3.5, we show a push-forward formula in equivariant elliptic cohomology which previously was not known in the present generality, although a form of it can be found in [Ga12].

3.1. Construction of oriented derived elliptic curves. Now we specify the set-up in which we would like to apply Lurie's machinery. Let $S = \text{Spec } R$ be an arbitrary (classical) affine scheme, with an elliptic curve $E \rightarrow S$ over S . Let $0 : S \rightarrow E$ be the zero section, Ω_E is the sheaf of relative Kähler differentials on E , and $\omega = 0^*(\Omega_E)$.

Definition 3.1. [GKV95, (1.8.1)] A local coordinate of E consists the following data:

- (1) a line bundle Π on E ;

- (2) an isomorphism $0^*\Pi \xrightarrow{\sim} 0^*\Omega_E$;
- (3) a rational section \mathfrak{l} of the bundle Π , such that \mathfrak{l} vanishes along the zero section $0(S)$, and under the map $d : \Pi \rightarrow \Pi \otimes \Omega_E$ we have $d(\mathfrak{l})|_{0(S)} = \text{id}_\Pi|_{0(S)}$ via the identification $0^*\Pi \cong 0^*\Omega_E$.

Note that ω is a locally free sheaf of modules of rank one over R . The rational section \mathfrak{l} of Π induces a formal group law on $\bigoplus_i H^0(S, \omega^{\otimes i})$, which is a power series $F(x, y) = x + y + \sum_{i+j \geq 2} a_{ij} x^i y^j$ with $a_{ij} \in H^0(S, \omega^{\otimes i+j-1})$, determined by the equality

$$\mathfrak{l}(u + v) = F(\mathfrak{l}(u), \mathfrak{l}(v))$$

in the formal completion of E along the zero section. Here the identification of the formal completion of E with $\bigoplus_i H^0(S, \omega^{\otimes i})[[u, v]]$ is given by the fixed isomorphism $0^*\Pi \rightarrow 0^*\Omega_E$.

Proposition 3.2. *Assume F is Landweber exact (e.g., see [Lu09, p.6]). Then there is an even and periodic E_∞ -ring spectrum A such that $\pi_{2n}A \cong \omega^{-n}$, and there is an oriented A -group \mathbb{E} with underlying scheme E and the orientation $\omega^{-1} \rightarrow \pi_2 A \cong \omega^{-1}$ is the identity map.*

The oriented A -group \mathbb{E} is called a derived elliptic curve. The proposition follows directly from a theorem due to Goerss, Hopkins, and Miller, reformulated in [Lu09, Theorem 1.1].

Proof. By base change, the elliptic curve $E \rightarrow \text{Spec } R$ extends to an elliptic curve on $\bigoplus_{n \in \mathbb{Z}} \omega^n$. This family endows $\text{Spec}(\bigoplus_{n \in \mathbb{Z}} \omega^n)$ with a map to $\mathcal{M}_{1,1}$, the (open) moduli stack of elliptic curves. This map is flat if and only if F is Landweber exact. Therefore, by [Lu09, Theorem 1.1], there is an even and periodic E_∞ -ring spectrum A representing a multiplicative cohomology theory whose coefficient ring $\pi_* A$ is $\bigoplus_{n \in \mathbb{Z}} \omega^n$. The derived elliptic curve is obtained in a similar way by composing the map $E \rightarrow \bigoplus_{n \in \mathbb{Z}} \omega^n$ and $\text{Spec}(\bigoplus_{n \in \mathbb{Z}} \omega^n) \rightarrow \mathcal{M}_{1,1}$. \square

Remark 3.3. (1) The Landweber exactness assumption is always satisfied, for example, when $R = \pi_0 A$ is an algebra over a field of characteristic zero.

(2) The ring spectrum A obtained this way is automatically periodic with periodicity given by $0^*\mathfrak{l} \in H^0(S, \omega) = \pi_2 A$.

(3) The structure of the homotopy commutative ring spectrum is easy to describe in concrete terms. Let $U \subset E$ be an affine open subset. Then on $\bigoplus_{i \in \mathbb{Z}} (\mathcal{O}_U \otimes_{\mathcal{O}_S} \omega^i)$ there is a formal group law coming from that on $\bigoplus_{i \in \mathbb{Z}} \omega^i$, which is Landweber exact. Therefore, by Goerss, Hopkins, and Miller's interpretation of Landweber exactness theorem, the graded ring $\bigoplus_{i \in \mathbb{Z}} (\mathcal{O}_U \otimes_{\mathcal{O}_S} \omega^i)$ is the homotopy group of some commutative ring spectrum. One can easily check that this construction is compatible with localization to principal affine open subsets of U , hence we obtain a sheaf of ring spectra on E .

In the remaining part of this paper, we always assume the ring spectrum A comes from a local coordinate of a classical elliptic curve over A , and the oriented derived elliptic curve \mathbb{E} is given by Proposition 3.2.

3.2. Equivariant elliptic cohomology theory. For any connected compact Lie group G , let G^{alg} be the corresponding algebraic reductive group with maximal algebraic torus $T^{\text{alg}} \supset T$. There is a derived scheme $\mathfrak{A}_G^{\text{der}}$, whose underlying scheme \mathfrak{A}_G is the moduli space of stable topologically trivial G^{alg} -bundles on E^\vee . For example, if $G = T$ is a torus of rank n , then $\mathfrak{A}_T^{\text{der}} = \mathbb{E}^n$.

we consider \mathbb{E} and A satisfying the following.

Assumption 3.4. For each connected compact Lie group G , there is a contravariant functor \mathcal{E}_G from the category of G -spaces to the category of quasi-coherent sheaves on $\mathfrak{A}_G^{\text{der}}$, satisfying the following properties:

- (1) \mathcal{E}_G maps homotopy equivalences to equivalences of quasi-coherent sheaves.
- (2) \mathcal{E}_G maps homotopy colimits of G -spaces to homotopy limits of quasi-coherent sheaves.
- (3) For an embedding of compact Lie groups $\phi : H \hookrightarrow G$, and finite H -space X , we have the following equivalence

$$\mathrm{Ind}_H^G : \mathcal{E}_G((X \times G)/H) \cong \phi_{\mathfrak{A}*} \mathcal{E}_H(X),$$

where $\phi_{\mathfrak{A}} : \mathfrak{A}_H^{\mathrm{der}} \rightarrow \mathfrak{A}_G^{\mathrm{der}}$ is induced by the embedding.

- (4) Let $\phi : H \rightarrow G$ be a group homomorphism, and let X be a G -space. Then X also has an action of H via ϕ . Let $\phi_{\mathfrak{A}} : \mathfrak{A}_H^{\mathrm{der}} \rightarrow \mathfrak{A}_G^{\mathrm{der}}$ be the induced morphism. Then we have a canonical isomorphism $\phi_{\mathfrak{A}}^*(\mathcal{E}_G(X)) \cong \mathcal{E}_H(X)$.
- (5) When $G = T$ is a torus, \mathcal{E}_T is the same functor in Theorem 2.3.

Remark 3.5. (1) It follows from Assumption 3.4.(3) that $\mathcal{E}_G(\mathrm{pt}) = \mathcal{O}_{\mathfrak{A}_G^{\mathrm{der}}}$.

- (2) The existence of the functors \mathcal{E}_G for (non-abelian) compact Lie group G is announced in [Lu09, § 5.1]. The properties (1), (2), and (3), on the level of global sections, are stated in [Lu09, Proposition 3.3].
- (3) If $\pi_0 A$ is a \mathbb{Q} -algebra, for any connected compact Lie group G and any finite G -space X , we define $\mathcal{E}_G(X)$ to be the W -invariants in $\pi_* \mathcal{E}_T(X)$ where $\pi : \mathfrak{A}_T^{\mathrm{der}} \rightarrow \mathfrak{A}_G^{\mathrm{der}}$ is the quotient map. Then, the assignment sending a finite G -space X to $\mathcal{E}_G(X)$ satisfies Assumption 3.4. Indeed, the only possibly non-trivial statement is Assumption 3.4(3), the proof of which can be found in [Ga12, Theorem 4.6].

It is reasonable to expect that Assumption 3.4 is satisfied for any oriented elliptic curve, but this statement is not in the literature. Proving this is beyond the scope of this paper. However, in the setting where we apply equivariant elliptic cohomology, $\pi_0 A$ is always a \mathbb{Q} -algebra. Therefore, by Remark 3.5(3), Assumption 3.4 is always satisfied.

As is shown in [GKV95], Assumption 3.4(3), which is a property about induction, implies the following property of change of groups. (See also [Ga12, Proposition 5.1].)

Corollary 3.6. *Under Assumption 3.4, let X be a finite G -space, and let K be a normal subgroup of G such that K acts on X freely. Then*

$$\mathcal{E}_G(X) \cong \phi_{\mathfrak{A}}^* \mathcal{E}_{G/K}(X/K)$$

where $\phi_{\mathfrak{A}} : \mathfrak{A}_H^{\mathrm{der}} \rightarrow \mathfrak{A}_G^{\mathrm{der}}$ is induced by the quotient $\phi : G \rightarrow G/K$.

Proof. By cellular induction, we only need to show this when $X = G/H$ for some closed subgroup $H < G$. The assumption that K acts freely on X implies that the natural map $\psi : H \rightarrow G \rightarrow G/K$ is an embedding. We have $(G/H)/K \cong (G/K)/H$. Hence, by Assumption 3.4(3), we obtain

$$\mathcal{E}_{G/K}((G/H)/K) \cong \mathcal{E}_{G/K}((G/K)/H) \cong \psi_{\mathfrak{A}*} \mathcal{E}_H(\mathrm{pt})$$

, where $\psi_{\mathfrak{A}} : \mathfrak{A}_H \rightarrow \mathfrak{A}_{G/K}$ is the natural embedding. Applying Assumption 3.4(3) again to the embedding $H < G$ and the H -space pt , we conclude the proof. \square

For any G -space X , we define $\mathcal{E}_G^*(X) = \bigoplus_{i \in \mathbb{Z}} \mathcal{E}_G^i(X)$, a sheaf of graded commutative algebras on the scheme \mathfrak{A}_G with $\mathcal{E}_G^i(X) = \pi_i(\mathcal{E}_G(X))$. In particular, $\mathcal{E}_G^0(X)$ is a quasi-coherent sheaf of commutative algebras on \mathfrak{A}_G . Define

$$\mathfrak{A}_G^X = \underline{\mathrm{Spec}}_{\mathfrak{A}_G} \mathcal{E}_G^0(X)$$

which is a scheme over \mathfrak{A}_G . The structure morphism $\mathfrak{A}_G^X \rightarrow \mathfrak{A}_G$ is denoted by π_X^G , or simply π_X if G is understood from the context.

Example 3.7. Take $\Pi = \Omega_E$, then a local coordinate of E amounts to a rational section \mathfrak{l} of Ω_E . In this case, we can identify $\mathcal{E}_G^i(X)$ with $\mathcal{E}_G^{i-2}(X) \otimes \omega^{-1}$. If $X \cong \mathbb{C}^n$, then $\mathcal{E}_G^i(X) = 0$ for odd i .

3.3. The GKV-classifying maps. We recall the GKV-classifying map defined in [GKV95, (1.6)]. Let X be a G -space. For any G -vector bundle V of rank n , let $Fr \rightarrow X$ be the associated frame bundle. The group $G \times U_n$ acts on Fr , so $\mathfrak{A}_{G \times U_n}^{Fr}$ is a scheme over $\mathfrak{A}_{G \times U_n}$, which in turn is a scheme over $\mathfrak{A}_{U_n} \cong E^{(n)}$. By Corollary 3.6, there is an isomorphism $\mathfrak{A}_{G \times U_n}^{Fr} \cong \mathfrak{A}_G^X$. These maps fit into the following commutative diagram

$$\begin{array}{ccccc} \mathfrak{A}_{G \times U_n}^{Fr} & \longrightarrow & \mathfrak{A}_{G \times U_n} & \longrightarrow & \mathfrak{A}_{U_n} \\ \downarrow \cong & & \downarrow & & \\ \mathfrak{A}_G^X & \longrightarrow & \mathfrak{A}_G & & \end{array}$$

The induced map $c_V : \mathfrak{A}_G^X \rightarrow \mathfrak{A}_{U_n}$ is called the *GKV-classifying map*.

Example 3.8. Let $G = \{1\}$, and $X = \mathbb{P}^1$ with the tautological line bundle $L = \mathcal{O}(-1)$ on it, so that the total space, with the zero section removed, is isomorphic to $\mathbb{A}^2 - \{0\}$. Let S^1 act on $\mathbb{A}^2 - \{0\}$ by weight $n \in \mathbb{Z}_{\geq 0}$, so that L is an S^1 -line bundle with respect to the trivial S^1 -action on \mathbb{P}^1 . This line bundle will be denoted by ϵ_n . We would like to see for different n what the GKV-classifying maps $c_{\epsilon_n} : \mathfrak{A}_1^{\mathbb{P}^1} \rightarrow \mathfrak{A}_{S^1} \cong E$ are.

By construction, we have $\mathfrak{A}_1^{\mathbb{P}^1} \cong \mathfrak{A}_{S^1}^{\mathbb{A}^2 - \{0\}} = \text{Spec}(\mathcal{E}_{S^1}^0(\mathbb{A}^2 - \{0\}))$. By Corollary 2.8, the stalk of $\mathcal{E}_{S^1}^0(\mathbb{A}^2 - \{0\})$ is zero away from the origin, and is nilpotent at the origin, hence the stalk is isomorphic to its completion at the origin. In particular, when $n = 1$, the map c_{ϵ_1} identifies $\mathfrak{A}_{S^1}^{\mathbb{A}^2 - \{0\}}$ with the Zariski tangent space of E at the origin; and when $n = 0$, c_{ϵ_0} is constant.

As in [GKV95, (1.9)], one can define the characteristic classes in equivariant elliptic cohomology as follows. Let $P < U_n$ be the parabolic subgroup such that $U_n/P = \mathbb{P}^n$. Then the projectivization $\mathbb{P}(V)$ is isomorphic to Fr/P . By Corollary 3.6, there is an isomorphism $\mathfrak{A}_{G \times P}^{Fr} \cong \mathfrak{A}_G^{\mathbb{P}(V)}$, which is a scheme over $\mathfrak{A}_P \cong \mathfrak{A}_{U_{n-1} \times S^1} \cong E^{(n-1)} \times E$.

Lemma 3.9. *We have a commutative diagram*

$$\begin{array}{ccccccc} \mathfrak{A}_G^{\mathbb{P}(V)} & \xrightarrow{\cong} & \mathfrak{A}_{G \times P}^{Fr} & \longrightarrow & \mathfrak{A}_P & \xrightarrow{\cong} & E^{(n-1)} \times E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathfrak{A}_G^X & \xrightarrow{\cong} & \mathfrak{A}_{G \times U_n}^{Fr} & \xrightarrow{c_V} & \mathfrak{A}_{U_n} & \xrightarrow{\cong} & E^{(n)}, \end{array}$$

where the right vertical map is the symmetrization defined in Section 1. Moreover,

- (1) the middle square is Cartesian;
- (2) the inclusion $\mathfrak{A}_G^{\mathbb{P}(V)} \cong \mathfrak{A}_G^X \times_{E^{(n)}} (E^{(n-1)} \times E) \subseteq \mathfrak{A}_G^X \times E$ is the embedding of a divisor.

The map $\mathfrak{A}_G^{\mathbb{P}(V)} \rightarrow E^{(n-1)} \times E$ will be denoted by $c_{\mathbb{P}(V)}$, and called the *GKV projective classifying map*.

Proof. Claim (1) is clear.

The natural map $\mathfrak{A}_G^X \times_{E^{(n)}} (E^{(n-1)} \times E) \subseteq \mathfrak{A}_G^X \times E$ is the pull-back of the natural map $\oplus \times \text{id} : E^{(n-1)} \times E \rightarrow E^{(n)} \times E$. The later map is a well-defined embedding of a divisor, hence, so is the former one. \square

3.4. The Thom isomorphism Theorem and Chern classes.

Definition 3.10. [GKV95, § 2.1] Let X be a finite G -space, and V a G -vector bundle. The subset $\mathfrak{A}_G^{\mathbb{P}(V)} \subset \mathfrak{A}_G^X \times E$ is a divisor. Let $\mathcal{O}(-\mathfrak{A}_G^{\mathbb{P}(V)})$ be its ideal sheaf. Define $\Theta_G(V)$ to be the line bundle on \mathfrak{A}_G^X which is the pull-back of $\mathcal{O}(-\mathfrak{A}_G^{\mathbb{P}(V)})$ along the map $\mathfrak{A}_G^X \rightarrow \mathfrak{A}_G^X \times E, x \mapsto (x, 0)$. If G is understood, we simply abbreviate $\Theta_G(V)$ as $\Theta(V)$.

Example 3.11. When $G = T$ and $X = \text{pt}$, then $\Theta(V)$ is isomorphic to $\mathcal{E}_T^0(Th(V))$ as quasi-coherent sheaves on \mathfrak{A}_T , which is also equal to $\mathcal{O}(-\ker \chi_{\lambda_V})$ defined in Section 1, where $\lambda_V \in \mathbb{Z}[\mathbb{X}^*(T)]$ is the character of the T -representation V .

Example 3.12. When $G = U_n$, $X = \text{pt}$, and the vector bundle ξ_n is the standard n -dimensional representation of U_n , by Lemma 1.4, $\Theta(\xi_n) = \mathcal{O}(-C)$. Moreover, $\vartheta^{U_n} = \prod_{i=1}^n \vartheta(x_i)$ is the natural section of $\Theta(\xi_n)^\vee$.

Proposition 3.13. For arbitrary G -space X ,

- (1) for any rank- n G -vector bundle V on X , we have $\Theta_G(V) \cong c_V^* \Theta_{U_n}(\xi_n)$, and under this isomorphism, $c_V^{-1} \vartheta^{U_n}$ is identified with the natural section of $\Theta_G(V)^\vee$.
- (2) For any two G -vector bundles V_1 and V_2 , we have a natural isomorphism

$$\Theta(V_1 \oplus V_2) \cong \Theta(V_1) \otimes \Theta(V_2);$$

- (3) The assignment sending any G -vector bundle V to the line bundle $\Theta(V)$ on \mathfrak{A}_G^X extends to a group homomorphism $\Theta : K_G(X) \rightarrow \text{Pic}(\mathfrak{A}_G^X)$.

Proof. Claim (1) follows from the definition.

Claim (2) follows directly from Lemma 1.5 and Remark 3.13.

Claim (3) follows from (2). □

For any G -vector bundle V , let $Th(V) = D(V)/S(V)$ be the Thom space, which is a finite G -space over X . The group G acts on $Th(V)$, hence $\mathfrak{A}_G^{Th(V)}$ is a scheme over \mathfrak{A}_G . The following is the Thom isomorphism in the equivariant elliptic cohomology.

Theorem 3.14 ([GKV95], (2.1.3)). Let V be an G -vector bundle. There is a canonical isomorphism $\pi_{X*} \Theta(V) \cong \mathcal{E}_G^0(Th(V))$ making the following diagram commutative

$$\begin{array}{ccc} \pi_{X*} \Theta(V) & \longrightarrow & \pi_{X*} \mathcal{O}_{\mathfrak{A}_G^X} \\ \downarrow \cong & & \parallel \\ \mathcal{E}_G^0(Th(V)) & \longrightarrow & \mathcal{E}_G^0(X). \end{array}$$

Here $\mathcal{E}_G^0(Th(V)) \rightarrow \mathcal{E}_G^0(X)$ is the pull-back via the embedding $X \rightarrow Th(V)$.

As a corollary, we have the following property about change of groups.

Corollary 3.15. Let X be a finite G -space, with a G -vector bundle V . Let $\phi : H \rightarrow G$ be a group homomorphism, and let $\phi_{\mathfrak{A}^X} : \mathfrak{A}_H^X \rightarrow \mathfrak{A}_G^X$ be the induced morphism. Then there is an isomorphism $\Theta_H(V) \cong \phi_{\mathfrak{A}^X}^* \Theta_G(V)$ of line bundles on \mathfrak{A}_H^X .

Let $q : X \rightarrow Y$ be a proper morphism between two smooth G -manifolds, and let $T\phi$ be the virtual vector bundle $q^*TY - TX$ on X . There is a push-forward morphism $q_{\sharp} : \Theta(Tq) \rightarrow q_{\mathfrak{A}*} \mathcal{E}_G^0(Y)$ of sheaves on \mathfrak{A}_G^X (see also [GKV95, § 2.3]). By adjunction, we also have the morphism $q_{\mathfrak{A}*} \Theta(Tq) \rightarrow \mathcal{E}_G^0(Y)$, still denoted by q_{\sharp} , which is characterized by the following two properties:

- (1) For any G -equivariant regular embedding $q : X \rightarrow Y$, we have $N_X Y = Tq$, and the push-forward is the composition of the Thom isomorphism $\mathcal{E}_G^0(X) \otimes \Theta(Tq) \cong \mathcal{E}_G^0(Th(q))$ and the pull-back $\mathcal{E}_G^0(Th(q)) \rightarrow \mathcal{E}_G^0(Y)$, where $Th(q)$ is the homotopy cofiber of the map $q : X \rightarrow Y$.
- (2) Let V be a representation of G and let SV be its unit sphere. The push-forward of the natural projection $q : SV \times X \rightarrow X$ is induced by $\mathcal{E}_G^*(SV \times X) \cong \Theta(V) \otimes \mathcal{E}_G^*(X)$.

As the push-forward is defined up to a twist by an explicit line bundle, we have two different notions of Chern classes in equivariant elliptic cohomology, coming from the orientation of the elliptic curve and the Thom isomorphism respectively. In the terminology of [GKV95], they are called the Chern classes and the Euler class, respectively.

Let f be any rational section of Ω_E on E (not necessarily a local coordinate of E). For any G -vector bundle V , define the i -th f -Chern class $c_i^f(V)$ of V to be $c_V^{-1}(e_i(f))$ as a rational section of $\mathcal{E}_G^0(X) \otimes \omega^{-i} \cong \mathcal{E}_G^{2i}(X)$. Here for any $i \geq 0$, e_i is the i -th elementary symmetric function, and $e_i(f)(x_1, \dots, x_r) := e_i(f(x_1), \dots, f(x_r))$ is a section of the line bundle $(\Pi^\vee)^{\otimes -i}$ on $E^{(r)}$.

On the other hand, define the Euler class

$$e(V) = c_V^{-1}(\vartheta^{U_n}) \in H^0(\mathfrak{A}_G^X, \Theta(V)^\vee) \cong H^0(\mathfrak{A}_G^X, c_V^* \Theta_{U_n}(\xi_n)^\vee).$$

It follows from the Thom isomorphism Theorem that $e(V)$ is equal to $z_\# : \Theta(V) \rightarrow \mathcal{E}_G^0(X)$ where $z : X \rightarrow V$ is the zero section.

Example 3.16. If the G -action on X is free, then $e(V)$ is the usual theta function. On the other hand, if the G -action on X is trivial, then $e(V) = c_1^l(V)$ where (Π, l) is a local coordinate of E in the sense of Definition 3.1.

Lemma 3.17 ([GKV95], (2.9.2)). *Let T be a torus, and let $a \in \mathfrak{A}_T$ be an R -point. Let $T(a) < T$ be as in (1), and let*

$$i_{a*} : \pi_{X^{T(a)*}} \Theta(T(X^{T(a)}))^\vee \rightarrow \pi_{X*} \Theta(TX)^\vee$$

be the push-forward induced by the inclusion $i_a : X^{T(a)} \rightarrow X$. Then, i_{a} is given by multiplication by $e(T_{X^{T(a)}} X)$, which is compatible with the push-forward induced by the inclusion $T(a) \rightarrow T(a')$ for $a, a' \in \mathfrak{A}_T$, and is invertible on a Zariski open subset of \mathfrak{A}_T containing a .*

3.5. The Quillen-Weyl-Kac formula. To illustrate the Thom bundles and the push-forwards, we prove the formula for push-forward in equivariant elliptic cohomology from a projective bundle, which will be used later. Let $V \rightarrow X$ be a rank- n G -vector bundle. Let $p : \mathbb{P}(V) \rightarrow X$ be the projection. Recall that

$$\begin{array}{ccc} \mathfrak{A}_G^{\mathbb{P}(V)} & \xrightarrow{c_{\mathbb{P}(V)}} & \mathfrak{A}_P \\ \downarrow & & \downarrow \\ \mathfrak{A}_G^X & \xrightarrow{c_V} & \mathfrak{A}_{U_n} \end{array}$$

is a Cartesian square. We describe the Thom bundle $\Theta(Tp)$ of the relative tangent bundle as follows. Let $c_{Tp} : E^{(n-1)} \times E \rightarrow E^{(n-1)}$ be the map $((y_1, \dots, y_{n-1}), y_n) \mapsto (y_1 - y_n, \dots, y_{n-1} - y_n)$. The pullback of the natural section $\prod_{i=1}^{n-1} \vartheta(x_i)$ of the line bundle $\Theta(\xi_{n-1})^\vee$ is equal to $\prod_{i=1}^{n-1} \vartheta(y_i - a)$, as a section of $c_{Tp}^* \Theta(\xi_{n-1})^\vee$. The Euler class $e(Tp)$ is the section $c_{\mathbb{P}(V)}^{-1}(\prod_{i \in [1, n-1]} \vartheta(y_i - y_n))$ of the line bundle $\Theta(Tp)^\vee \cong c_{\mathbb{P}(V)}^* c_{Tp}^* \Theta(\xi_{n-1})^\vee$. As sheaves on $E^{(n)}$ we have $\mathcal{E}_G^0(\mathbb{P}(V)) \cong \mathcal{E}_G^0(X) \otimes_{\mathcal{O}_{E^{(n)}}} \mathcal{O}(E^{(n-1)} \times E)$. So we can write any local section f of $\mathcal{E}_G^0(\mathbb{P}(V))$ as a function on E valued in

$\mathcal{E}_G^0(X)$. By Theorem 2.4, the push-forward $p_{\sharp} : \mathcal{E}_G^0(\mathbb{P}(V)) \rightarrow p_{\sharp}^* \mathcal{E}_G^0(X) \otimes \Theta(Tp)^{\vee}$ is given by

$$f \mapsto \frac{f(-y_n)}{\prod_{i=1}^{n-1} c_{\mathbb{P}(V)}^{-1} \vartheta(y_i - y_n)}.$$

To summarize, we have the following Quillen-Weyl-Kac formula.

Proposition 3.18. *Let $V \rightarrow X$ be a rank- n G -vector bundle with $p : \mathbb{P}(V) \rightarrow X$ the projection. The push-forward $p_{\sharp} : p_{\sharp*} \Theta(Tp) \rightarrow \mathcal{E}_G^0(X)$ is given by*

$$(2) \quad \sum_{i=1}^{n-1} (i, n) \frac{f(-y_n)}{\prod_{j=1}^{n-1} c_{\mathbb{P}(V)}^{-1} \vartheta(y_j - y_n)}$$

for any local section f of $p_{\sharp*} \Theta(Tp)$. Here $(i, n) \in \mathfrak{S}_n$ is the transposition.

An illustrating example of this formula will be given in Lemma 5.3.

Let $G = \mathrm{SU}_2 \subset G^{\mathbb{C}} = \mathrm{PGL}_2(\mathbb{C})$ acting on \mathbb{A}^2 in the natural way. We identify \mathbb{P}^1 with $\mathbb{P}(\mathbb{A}^2)$, and T with S^1 whose action on \mathbb{A}^2 has weights $-\alpha/2$ and $\alpha/2$. The projection $\mathbb{P}^1 \rightarrow \mathrm{pt}$ will be denoted by p . Let $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the i -th projection, for $i = 1, 2$.

Lemma 3.19. *We have the following formula for $p_{1\sharp} p_2^{\sharp} : \Theta(Tp) \rightarrow \mathcal{O}_{\mathbb{P}^1}$:*

$$p_{1\sharp} p_2^{\sharp}(\sigma) = \frac{s_{\alpha} \sigma(x)}{\vartheta(-\chi_{\alpha})} + \frac{\sigma(x)}{\vartheta(\chi_{\alpha})}.$$

Proof. Write the coordinate of $\mathfrak{A}_G(\mathbb{P}(\mathbb{A}^2)) \cong \mathfrak{A}_T(\mathrm{pt}) \cong E$ as (x) . The s_{α} -action on $\mathfrak{A}_G(\mathbb{P}(\mathbb{A}^2))$ sends (x) to $(-x)$. The fact that the T -action on \mathbb{A}^2 has weights $-\alpha/2$ and $\alpha/2$ implies that the GKV projective classifying map $c_{\mathbb{P}(\mathbb{A}^2)} : E \rightarrow E \times E$ sends x to $(-x, x)$.

Recall that we have a natural isomorphism

$$\mathfrak{A}_G(\mathbb{P}^1) \cong E \rightarrow (E/\mathfrak{S}_2) \times_{E^{(2)}} (E \times E)$$

with the map $E \rightarrow (E/\mathfrak{S}_2) \times (E \times E)$ given by $x \mapsto (x, (-x, x))$. By our convention, we write the coordinates of the $E \times E$ -factor as (y_1, y_2) . We consider any local section σ of $\mathcal{E}_T^0(\mathbb{P}^1)$ on $(E/\mathfrak{S}_2) \times (E \times E)$ as a $\mathcal{O}((E/\mathfrak{S}_2) \times E)$ -valued function $\sigma(y_2)$ in the variable $y_2 \in E$. Note that the map $E \rightarrow (E/\mathfrak{S}_2) \times_{E^{(2)}} (E \times E)$ induces an isomorphism from E to the last E -factor in $(E/\mathfrak{S}_2) \times (E \times E)$. In particular, in the Quillen-Weyl-Kac formula y_2 can be substituted by x , and y_1 can be substituted by $-x$. The Quillen-Weyl-Kac formula (2) yields

$$\begin{aligned} p_{1\sharp} \left(p_2^{\sharp} \sigma \right) &= \left(\frac{\sigma(-x)}{\vartheta(x_{-\alpha/2} - \chi_{\alpha/2})} + \frac{\sigma(x)}{\vartheta(\chi_{\alpha/2} - x_{-\alpha/2})} \right) \\ &= \left(\frac{s_{\alpha} \sigma(x)}{\vartheta(-\chi_{\alpha})} + \frac{\sigma(x)}{\vartheta(\chi_{\alpha})} \right). \end{aligned}$$

□

3.6. Convolution with Lagrangian correspondences. Now we briefly recall that convolution with Lagrangian correspondences defines an associative algebra. For details, we refer to [GKV95, § 2.8].

Let M_i be smooth G -manifolds, for $i = 1, 2$. Let $N_i = T^*M_i$, endowed with G -action such that $N_i \rightarrow M_i$ is equivariant. Let $Z \subseteq T^*(M_1 \times M_2)$ be a regular embedding of G -invariant subvariety, not necessarily smooth or Lagrangian. With notations

$$M_1 \xleftarrow{b_1} T^*M_1 \xleftarrow{\mathrm{pr}_1} Z \xrightarrow{\mathrm{pr}_2} T^*M_2 \xleftarrow{i_2} M_2,$$

then the line bundle

$$\Xi_G(Z) := \Theta(\mathrm{pr}_2) \otimes \Theta(\mathrm{pr}_1^* b_1^* N_1) \otimes \Theta(\mathrm{pr}_2^* b_2^* N_2)^{-1}$$

on \mathfrak{A}_G^Z defines a map

$$\pi_{Z*} \Xi_G(Z) \rightarrow \mathcal{H}\mathrm{om}_{\mathfrak{A}_G}(\pi_{M_1*} \Theta(N_1)^{-1}, \pi_{M_2*} \Theta(N_2)^{-1}).$$

Lemma 3.20. *Then $\Xi_G(Z) \cong \Theta_Z(N_1 \times N_2) \otimes \Theta(\mathrm{pr}_1^* b_1^* N_1)^{-1} \otimes \Theta(\mathrm{pr}_2^* b_2^* N_2)^{-1}$*

Proof. By definition, pr_2 is the composition $i : Z \hookrightarrow N_1 \times N_2$ and $\mathrm{pro}_2 : N_1 \times N_2 \rightarrow N_2$. Hence

$$\Theta(\mathrm{pr}_2) \cong \Theta_Z(N_1 \times N_2) \otimes \Theta(\mathrm{pr}_1^* T N_1)^{-1}.$$

We have that $T N_1$ is an extension between $T M_1$ and N_1 , considered as G -vector bundles on M_1 . Therefore, $\Theta(T N_1)^{-1} \cong \Theta(T M_1)^{-1} \otimes \Theta(N_1)^{-1}$. Plug it in to the definition of $\Xi_G(Z)$, we get the conclusion. \square

Lemma 3.21 ([GKV95], (2.8.4), (2.8.5)). *Let M_3 be a smooth G -manifold, and $N_3 = T^* M_3$, and $Z' \subseteq N_2 \times N_3$ be a closed subvariety. Then*

(1) *The composition of actions lifts to a morphism*

$$\pi_{Z*} \Xi_G(Z) \otimes \pi_{Z'*} \Xi_G(Z') \rightarrow \pi_{Z \circ Z'*} \Xi_G(Z \circ Z')$$

which is associative in the usual sense.

(2) *When $M_1 = M_2 = M$ and $Z \circ Z = Z$, then $\pi_{Z*} \Xi_G(Z)$ is a sheaf of algebras and $\pi_{M*} \mathcal{E}_G^0(N)$ on \mathfrak{A}_G is a representation of it.*

As a consequence of Corollary 3.15, we have the following property about change of groups.

Corollary 3.22. *Notations as above, let $\phi : H \rightarrow G$ be a group homomorphism. Let $\phi_{\mathfrak{A}} : \mathfrak{A}_H \rightarrow \mathfrak{A}_G$ be the induced morphism. Then there is an isomorphism $\pi_{Z*}^H \Xi_H(Z) \cong \phi_{\mathfrak{A}}^*(\pi_{Z*}^G \Xi_G(Z))$, which commutes with convolution.*

In particular, if N_1 and N_2 are both endowed with trivial G -actions, then there is an isomorphism $\pi_{Z*}^T \Xi_T(Z) \cong p_{\mathfrak{A}_T}^*(\mathcal{E}(Z))$ where $p_{\mathfrak{A}_T}$ is the projection $\mathfrak{A}_T \rightarrow \mathrm{pt}$.

Let T be a torus, and let $a \in \mathfrak{A}_T$ be an R -point. Let $T(a) < T$ be as (1). The inclusion $i : Z^{T(a)} \hookrightarrow Z^T$ induces

$$(3) \quad i^{\sharp} : \Theta_T(N_1 \times N_2) \rightarrow i_{\mathfrak{A}*} \Theta_{T(a)}(N_1^{T(a)} \times N_2^{T(a)}).$$

Assumption 3.23. Let ξ be a 1-dimensional representation of G , and assume the G -action on the fiber direction of $T^* M_i$ is given by the induced action on M_i twisted by ξ . Assume Z is a Lagrangian subvariety.

As a consequence of Lemma 3.17, under Assumption 3.23, multiplication by $e(T_{M_1^{T(a)}} M_1)^{-1} \otimes e(T_{M_2^{T(a)}} M_2)^{-1}$ is a well-defined rational morphism of sheaves

$$(4) \quad \Theta(\mathrm{pr}_1^* b_1^* T^* M_1^{T(a)}) \otimes \Theta(\mathrm{pr}_2^* b_2^* T^* M_2^{T(a)}) \rightarrow \Theta(\mathrm{pr}_1^* b_1^* T M_1) \otimes \Theta(\mathrm{pr}_2^* b_2^* T M_2)$$

on \mathfrak{A}_T , which is regular and invertible on a Zariski open subset containing a .

Proposition 3.24 ([GKV95], (2.10.3)). *Under Assumption 3.23, let*

$$\rho_a : \pi_{Z*}^T \Xi_T(Z) \rightarrow \pi_{Z^{T(a)*}}^T \Xi_T(Z^{T(a)})$$

be the rational morphism which is the composition of (3) and (4). The following is true.

- (1) *There is a Zariski open subset of \mathfrak{A}_T containing a , the restriction of ρ_a on which is regular and invertible. In particular, ρ_a is invertible on the stalks over $a \in \mathfrak{A}_T$.*
- (2) *ρ_a commutes with the convolution.*

Let $\phi : T \rightarrow T/T(a)$ be the quotient and let $\phi_{\mathfrak{A}} : \mathfrak{A}_T \rightarrow \mathfrak{A}_{T/T(a)}$ be the induced map. By Corollary 3.22, we have an isomorphism

$$\pi_{Z^{T(a)*}}^T \Xi_{T(a)}(Z^{T(a)}) \cong \phi_{\mathfrak{A}}^* \pi_{Z^{T(a)*}}^{T/T(a)} \Xi_{T/T(a)}(Z^{T(a)}),$$

which commutes with convolution. When $T(a) = T$, $\mathfrak{A}_{T/T(a)}$ is $\text{Spec } R$ and $\Xi_{T/T(a)}(Z^{T(a)})$ is $\mathcal{E}_{\{1\}}^0(Z^{T(a)})$ as R -modules. Summarizing Corollary 3.22 and Proposition 3.24, we get the following bivariant Riemann-Roch theorem. Let R_a be the skyscraper sheaf at the R -point $a \in \mathfrak{A}_G$.

Theorem 3.25. *Assume $a \in \mathfrak{A}_T$ is such that $T(a) = T$. Then morphism*

$$\pi_{Z*} \Xi_G(Z) \otimes_{\mathcal{O}_{\mathfrak{A}_G}} R_a \rightarrow \mathcal{E}_1^0(Z^{T(a)}) \cong H_*(Z^{T(a)}; R)$$

induced by ρ_a is an isomorphism, and commutes with convolution. If $Z \circ Z = Z$, then the above map is an isomorphism of algebras.

Here, $H_*(Z^{T(a)}; R)$ is the Borel-Moore homology with R -coefficients, endowed with the convolution product, and the isomorphism $\mathcal{E}_1^0(Z^{T(a)}) \cong H_*(Z^{T(a)}; R)$ is the usual bivariant Riemann-Roch Theorem (see also [ZZ14, Corollary 5.8]).

4. THE ELLIPTIC AFFINE HECKE ALGEBRA: THE ALGEBRAIC CONSTRUCTION

In this section, we recall the definition of the elliptic affine Demazure algebra and the elliptic affine Hecke algebra. We prove some basic properties about the structures of these algebras. Throughout this section we assume that R is an integral domain, unless otherwise stated.

4.1. The elliptic affine Hecke algebra. We recall the notion of root datum following [SGA3, Exp. XXI, § 1.1]. A root datum is a lattice Λ , a non-empty $\Sigma \subset \Lambda$ subset, together with an embedding $\Sigma \hookrightarrow \Lambda^\vee$, $\alpha \mapsto \alpha^\vee$ where Λ^\vee is the dual of Λ . An element in Σ is called a root, and the sub-lattice Λ_r of Λ generated by Σ is called the root lattice. The set $\Lambda_w = \{w \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle w, \alpha^\vee \rangle \in \mathbb{Z}\}$ is called the weight lattice. One can similarly define the dual weight lattice Λ_w^\vee . The rank of Λ is called the rank of the root datum. A root datum is irreducible if it is not a direct sum of two non-trivial root data, and is called semi-simple if $\Lambda_r \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. We always assume that the root datum is semi-simple. A root datum is said to be adjoint (resp. simply connected) if $\Lambda = \Lambda_r$ (resp. $\Lambda = \Lambda_w$).

A subset $\Phi = \{\alpha_1, \dots, \alpha_n\} \subseteq \Sigma$ is called a set of simple roots if it is a basis of Λ_r . A choice of Φ leads to a decomposition of Σ into a disjoint union of negative and positive roots $\Sigma^- \cup \Sigma^+$. The automorphism $\Lambda \rightarrow \Lambda : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ determined by $\alpha \in \Sigma$ is called the simple reflection determined by α , denoted by s_α . The Weyl group W is the subgroup of $\text{Aut}(\Lambda)$ generated by all the simple reflections. For $\alpha_i \in \Phi$, denote $s_i = s_{\alpha_i}$. The Weyl group is also generated by $\{s_i \mid \alpha_i \in \Phi\}$. For each $w \in W$, define $\Sigma(w) = w\Sigma^- \cap \Sigma^+$.

We now recall the definition of the elliptic affine Hecke algebra given in [GKV97]. Let Γ be a free abelian group of rank one, and let E be an elliptic curve over a commutative ring R . Consider $\mathfrak{A} := E \otimes_{\mathbb{Z}} (\Gamma \oplus \Lambda^\vee) \cong E^{n+1}$, which is an abelian variety. Each group homomorphism $\lambda : \Gamma \oplus \Lambda^\vee \rightarrow \mathbb{Z}$ extends to a map of abelian varieties $\chi_\lambda : \mathfrak{A} \rightarrow E = E \otimes_{\mathbb{Z}} \mathbb{Z}$. In particular, each $\alpha \in \Sigma$ defines a map of abelian varieties χ_α . Let D^α be the divisor of \mathfrak{A} defined by the kernel of χ_α . Let $\gamma : \Gamma \oplus \Lambda^\vee \rightarrow \mathbb{Z}$ be a group homomorphism which is zero on Λ^\vee and isomorphism on Γ . Let $D^{\alpha, \gamma}$ be the divisor on \mathfrak{A} defined by the kernel of $\chi_\alpha - \chi_\gamma$. We have $D^\alpha = D^{-\alpha}$, and $w(D^\alpha) = D^{w^{-1}(\alpha)}$. The action of

the Weyl group W on Λ^\vee naturally extends to an action on \mathfrak{A} . Let $\pi : \mathfrak{A} \rightarrow \mathfrak{A}/W$ be the quotient, and $\mathcal{S} := \pi_* \mathcal{O}_{\mathfrak{A}}$ on \mathfrak{A}/W . There is a natural action of W on the sheaf \mathcal{S} . Define $\mathcal{S}_W := \mathcal{S} \rtimes W$, a coherent sheaf of associative algebras. Note that \mathcal{S} is a sheaf of modules over \mathcal{S}_W . On any open subset, sections of \mathcal{S}_W coming from elements in W will be denoted by δ_w . For any root $\alpha \in \Sigma$, we write $\delta_\alpha = \delta_{s_\alpha}$.

Let $\mathfrak{A}^c := (\mathfrak{A} \setminus (\cup_{\alpha \in \Sigma} D^\alpha))/W$, and let $j : \mathfrak{A}^c \hookrightarrow \mathfrak{A}/W$ be the inclusion. The action of \mathcal{S}_W on \mathcal{S} induces a morphism $\rho : \mathcal{S}_W|_{\mathfrak{A}^c} \rightarrow \mathcal{E}nd(\mathcal{S}|_{\mathfrak{A}^c})$. Applying j_* , we get a morphism

$$j_* \rho : j_*(\mathcal{S}_W|_{\mathfrak{A}^c/W}) \rightarrow j_* \mathcal{E}nd(\mathcal{S}|_{\mathfrak{A}^c/W}).$$

Let $p : \mathcal{E}nd(\mathcal{S}) \rightarrow j_* \mathcal{E}nd(\mathcal{S}|_{\mathfrak{A}^c})$ be the right adjoint to the natural map $j^* \mathcal{E}nd(\mathcal{S}) \rightarrow \mathcal{E}nd(\mathcal{S}|_{\mathfrak{A}^c})$.

Definition 4.1. We define $\widehat{\mathcal{D}}$ to be $(j_* \rho)^{-1}(p(\mathcal{E}nd(\mathcal{S})))$, as a quasi-coherent subsheaf of algebras of $j_*(\mathcal{S}_W|_{\mathfrak{A}^c/W})$.

We consider local sections of $j_*(\mathcal{S}_W|_{\mathfrak{A}^c})$ written as $\sum_{w \in W} f_w \delta_w$ where f_w are local sections of $j_*(\mathcal{S}|_{\mathfrak{A}^c/W})$, satisfying the following conditions.

- R1** for any root α and $w \in W$, each f_w has a pole of order at most one along the divisor D^α ;
- R2** for any root α and $w \in W$, the residues of f_w and $f_{s_\alpha w}$ along the divisor D^α differ by a negative sign;
- R3** for any $\alpha \in \Sigma(w)$, f_w vanishes along the divisor $D^{\alpha, \gamma}$.

Definition 4.2. [GKV97, Definition 1.3] The elliptic affine Demazure algebra \mathcal{D} is defined to be the quasi-coherent subsheaf of $j_*(\mathcal{S}_W|_{\mathfrak{A}^c})$ whose local sections satisfy conditions **R1** and **R2**. The elliptic affine Hecke algebra \mathcal{H} is defined to be the quasi-coherent subsheaf of \mathcal{D} whose local sections satisfy **R3**.

Remark 4.3. Fix a local uniformizer of E , which defines an elliptic formal group law F_e . Then the completion of the stalk of $\mathcal{O}_{\mathfrak{A}}$ at the origin of E is precisely the formal group algebra defined in [CPZ13, Definition 2.4], and the completion of \mathcal{D} and \mathcal{H} at the origin of \mathfrak{A}/W , are the elliptic formal affine Demazure algebra \mathbf{D}_{F_e} and the elliptic formal affine Hecke algebra \mathbf{H}_{F_e} considered in [ZZ14], respectively. For general formal group law F , the dual of \mathbf{D}_F is the algebraic model of the corresponding equivariant oriented cohomology of complete flag variety. For more details, please refer to [CZZ12], [CZZ13] and [CZZ14].

The following proposition will be referred to as the structure theorem of the elliptic affine Demazure algebra and the elliptic affine Hecke algebra.

Proposition 4.4. [GKV97, Theorem 4.4] *Let \mathcal{J} be the sheaf of ideals in \mathcal{S} corresponding to the divisor $\sum_{\alpha \in \Sigma} D^{\alpha, \gamma}$ on \mathfrak{A} . We have*

$$\mathcal{D} \cong \widehat{\mathcal{D}}$$

as sheaves of algebras on \mathfrak{A}/W , and \mathcal{H} is isomorphic to the subsheaf of algebras of \mathcal{D} consisting of sections u of \mathcal{D} such that

$$u(\pi_* \mathcal{J}) \subset \pi_* \mathcal{J}.$$

For any f rational section of Ω_E , which vanishes of order 1 along the zero section of E , define $\text{Sing}(f)$ to be the set of non-origin zeros of f union with the set of poles of f . Define

$$T_\alpha^f = \frac{f(\chi_\gamma)}{f(\chi_\alpha)} + (1 - \frac{f(\chi_\gamma)}{f(\chi_\alpha)})\delta_\alpha,$$

which is a regular section of \mathcal{H} on the open complement of $\chi_\gamma^{-1}(\text{Sing}(f)) \cup \chi_\alpha^{-1}(\text{Sing}(f))$.

Example 4.5 ([GKV97], § 4). Let E be a complex elliptic curve. Let x be an order-2 torsion point on E , and consider the Jacobi-sign function sn , which is the unique rational function that has simple zeros at the origin and x , poles of order 1 at the other two order-2 points, and derivative 1 at the origin. According to Definition 3.1, sn is a local coordinate of E . The sheaf of algebras \mathcal{S} is $\pi_* \mathcal{O}_{E^{n+1}}$. Let

$$V = [\mathfrak{A} \setminus (\cup_{\alpha \in \Sigma^+} \chi_\alpha^{-1}(x))] / W.$$

Then \mathcal{H} is the subalgebra of $\text{End}(H^0(V, \mathcal{S}))$ generated by $H^0(V, \mathcal{S})$ and $T_\alpha = \frac{\text{sn}(\chi_\alpha)}{\text{sn}(\chi_\alpha)} + (1 - \frac{\text{sn}(\chi_\alpha)}{\text{sn}(\chi_\alpha)})\delta_\alpha$ for $\alpha \in \Phi$.

4.2. Filtration by the Bruhat order. For any $w \in W$, define $\mathcal{H}_{\leq w} \subseteq \mathcal{H}$ to be the subsheaf whose local sections consist of $\sum_{y \leq w} f_y \delta_y$. Denote $\mathcal{H}_w = \mathcal{H}_{\leq w} / \mathcal{H}_{< w}$.

For any point $p \in \mathfrak{A}$, for any simple root α , let p_α be the α -coordinate of p . Let $f_{p,\alpha}$ be a rational function on E such that $p, -p \notin \text{Sing}(f_{p,\alpha})$. Note that such a function always exists (although non-unique). Let $U_{\alpha,p} \subseteq E$ be the open complement of $\text{Sing}(f_{p,\alpha}) \cup \text{Sing}(-f_{p,\alpha})$. Then $U := \prod_\alpha U_{\alpha,p}$ is a W -invariant open subset of \mathfrak{A} that contains p . This open subset U depends on p and $f_{\alpha,p}$.

Lemma 4.6. *For each $w \in W$, we fix a reduced sequence $I_w = (i_1, \dots, i_l)$ of w , i.e., $w = s_{i_1} \cdots s_{i_l}$.*

- (1) *The sheaf of rational sections of \mathcal{S} , whose local sections consisting of f that has simple poles at each divisor D^α , and vanishes along the divisors $D^{\alpha,\gamma}$ for each $\alpha \in \Sigma(w)$, is a locally free of rank 1 over \mathcal{S} .*
- (2) *On U/W , the sheaf in (1) is globally free, generated by $F_{I_w} := (1 - \frac{f_{\alpha_{i_1},p}(\chi_\gamma)}{f_{\alpha_{i_1},p}(\chi_{\alpha_{i_1}})}) \cdots (1 - \frac{f_{\alpha_{i_l},p}(\chi_\gamma)}{f_{\alpha_{i_l},p}(\chi_{\alpha_{i_l}})})$.*
- (3) *$\mathcal{H}_w|_{U/W}$ is free of rank 1 as a module over $\mathcal{S}|_{U/W}$, generated by $T_{I_w} := T_{\alpha_{i_1},p}^{f_{\alpha_{i_1},p}} \cdots T_{\alpha_{i_l},p}^{f_{\alpha_{i_l},p}}$.*

Proof. Claim (1) is clear. Claim (2) is a consequence of (1). An easy calculation as in [GKV97, Lemma 2.8.(ii)] shows that $T_{I_w} - F_{I_w} \delta_w \in \mathcal{H}_{< w}|_{U/W}$. Hence, (3) follows from (2). \square

The following theorem is essentially proved in [GKV95].

Theorem 4.7. *We have the following*

- (1) \mathcal{H}_w is a locally free sheaf of modules over \mathcal{S} of rank 1;
- (2) \mathcal{H} is a locally free sheaf of modules over \mathcal{S} of rank $|W|$;
- (3) \mathcal{H} is a locally free sheaf of modules over $\mathcal{O}_{\mathfrak{A}/W}$ of rank $|W|^2$.

Proof of Theorem 4.7. We prove, by induction on Bruhat order, that $\mathcal{H}_{\leq w}$ is locally free of rank $\#\{y \in W \mid y \leq w\}$. The rest of the theorem follows easily from this, where the inductive step is taken care of by Lemma 4.6.(3) above. \square

4.3. The elliptic Demazure operators. Recall that $\mathcal{O}(-\{0\})$ is the line bundle corresponding to the divisor $\{0\}$ on E , and $\vartheta(x)$ is the natural section of $\mathcal{O}(-\{0\})^\vee$. Let $\mathcal{L}_\alpha = \mathcal{O}(-D^\alpha)$, and let $\vartheta(\chi_\alpha)$ be the section of \mathcal{L}_α^\vee , which is the pull-back of ϑ along $\chi_\alpha : \mathfrak{A} \rightarrow E$. For each root α , we define

$$X_\alpha := \frac{1}{\vartheta(\chi_\alpha)} - \frac{1}{\vartheta(\chi_\alpha)} \delta_\alpha.$$

Considering $\vartheta(\chi_\alpha)$ as a section of $\pi_* \mathcal{L}_\alpha^\vee$, then X_α is a well defined element in

$$H^0(\mathfrak{A}/W, j_*(\mathcal{H} \text{om}(\mathcal{S}, \pi_* \mathcal{L}_\alpha^\vee)|_{\mathfrak{A}^c})).$$

For $\alpha_i \in \Phi$ we write $X_i = X_{\alpha_i}$ for short.

Lemma 4.8. *For any root α , the element $\Delta_\alpha := \rho(X_\alpha) \in \mathcal{H}\text{om}(\mathcal{S}, \pi_* \mathcal{L}_\alpha^\vee)|_{\mathfrak{A}^c/W}$ extends to a global section of $\mathcal{H}\text{om}(\mathcal{S}, \pi_* \mathcal{L}_\alpha^\vee)$.*

Proof. It suffices to show that for each local section σ of \mathcal{S} on an open set U , the element $\rho(X_\alpha)\sigma$ in $H^0(U, \pi_* \mathcal{L}_\alpha^\vee)$ is regular along the divisor $U \cap \pi(D^\alpha)$. This in turn amounts to show that on $U \cap \mathfrak{A}^c/W$, the rational section of $\pi_* \mathcal{L}_\alpha^\vee$

$$\frac{\sigma - s_\alpha(\sigma)}{\vartheta(\chi_\alpha)}$$

has numerator vanishing along the divisor $\pi(D^\alpha)$. But this is clear from definition. \square

Lemma 4.9. *The operators satisfies the following relations:*

- (1) $\delta_w X_\alpha \delta_{w^{-1}} = X_{w(\alpha)} \in \text{Hom}(\mathcal{S}, \pi_* \mathcal{L}_{w\alpha}^\vee)$;
- (2) for any open $U \subseteq \mathfrak{A}^c$ and $\sigma \in H^0(U, \mathcal{S})$, $X_\alpha \sigma - s_\alpha(\sigma) X_\alpha = \Delta_\alpha(\sigma)$.
- (3) $X_\alpha^2 = 0$.

Proof. (1) follows from the fact that $w(\vartheta(\chi_\alpha)) = \vartheta(\chi_{w(\alpha)})$. (2) follows from calculation similar as in the twisted formal group algebra case. (3) follows from the fact that $\vartheta(\chi_\alpha) = -\vartheta(\chi_{-\alpha})$. \square

4.4. The enlarged elliptic affine Demazure algebra. Recall

$$\tilde{\mathcal{S}} = \bigoplus_{\lambda \in \mathbb{Z}[\mathbb{X}^*(T)]} \mathcal{L}_\lambda.$$

Lemma 4.8 implies that X_α is a global section of $\mathcal{E}\text{nd}(\tilde{\mathcal{S}})$ for any $\alpha \in \Sigma$.

Similar to the situation of \mathcal{S} , we define $\tilde{\mathcal{S}}_W = \tilde{\mathcal{S}} \rtimes W$, and hence we have $\rho : \tilde{\mathcal{S}}_W \rightarrow \mathcal{E}\text{nd}(\tilde{\mathcal{S}})$. Let $p : \mathcal{E}\text{nd}(\tilde{\mathcal{S}}) \rightarrow j_* \mathcal{E}\text{nd}(\tilde{\mathcal{S}}|_{\mathfrak{A}^c})$ be the right adjoint to $j^* \mathcal{E}\text{nd}(\tilde{\mathcal{S}}) \rightarrow \mathcal{E}\text{nd}(\tilde{\mathcal{S}}|_{\mathfrak{A}^c})$. We define $\hat{\mathcal{D}}^e$ to be $(j_* \rho)^{-1} \left(p(\mathcal{E}\text{nd}(\tilde{\mathcal{S}})) \right)$, as a quasi-coherent subsheaf of algebras of $j_*(\tilde{\mathcal{S}}_W|_{\mathfrak{A}^c})$, where $j_* \rho : j_*(\mathcal{S}_W|_{\mathfrak{A}^c}) \rightarrow j_* \mathcal{E}\text{nd}(\mathcal{S}|_{\mathfrak{A}^c})$ is j_* applied to ρ .

We define \mathcal{D}^e to be the subsheaf of $j_*(\tilde{\mathcal{S}}_W|_{\mathfrak{A}^c})$, whose local sections consist of $\sum_{w \in W} f_w \delta_w$ where f_w 's are local sections of $j_*(\tilde{\mathcal{S}}|_{\mathfrak{A}^c})$ satisfying conditions **R1** and **R2**. Clearly X_α is a global section of \mathcal{D}^e for any $\alpha \in \Sigma$. Similar argument as [GKV97, Theorem 4.4] shows that $\hat{\mathcal{D}}^e \cong \mathcal{D}^e$.

Definition 4.10. Let $\tilde{\mathcal{D}}^e$ be the subsheaf of $\mathcal{O}_{\mathfrak{A}/W}$ -algebras of $j_*(\tilde{\mathcal{S}}_W|_{\mathfrak{A}^c})$ on \mathfrak{A}/W , generated by the image of $\tilde{\mathcal{S}} \rightarrow j_*(\tilde{\mathcal{S}}_W|_{\mathfrak{A}^c})$ and X_i with $i = 1, \dots, n$.

By definition and Lemma 4.9, X_α is a global section of $\tilde{\mathcal{D}}^e$. Lemma 4.8 implies that there is a morphism $\tilde{\mathcal{D}}^e \rightarrow \mathcal{E}\text{nd}(\tilde{\mathcal{S}})$ of sheaves of algebras on \mathfrak{A}/W , hence, $\tilde{\mathcal{D}}^e$ is a subsheaf of algebras of $\hat{\mathcal{D}}^e$.

Theorem 4.11. *If R is an integral domain containing $1/2$, then $\hat{\mathcal{D}}^e = \tilde{\mathcal{D}}^e = \mathcal{D}^e$.*

Proof. This is a local property, so we can reduce to an open subset of \mathfrak{A}/W . Then $\hat{\mathcal{D}}^e = \tilde{\mathcal{D}}^e$ follows similarly as [ZZ14, Theorem 3.8]. \square

For any sequence $I = (i_1, \dots, i_k)$, we define

$$X_I = X_{i_1} \circ \dots \circ X_{i_k}.$$

We use I_w to denote a reduced sequence of $w \in W$. According to [BE90], this definition depends on the choice of I_w . Modifying the arguments in [CZZ12] and [ZZ14, § 3], we get the following properties:

Proposition 4.12. *If R is an integral domain containing $1/2$, we have the following properties.*

- (1) $\{X_{I_w}\}_{w \in W}$ is a basis of \mathcal{D}^e as a sheaf of left $\tilde{\mathcal{S}}$ -modules. In particular, \mathcal{D}^e is globally free as a sheaf of $\tilde{\mathcal{S}}$ -modules.
- (2) \mathcal{D}^e is generated as a sheaf of $\mathcal{O}_{\mathfrak{A}/W}$ -algebras locally generated by $\tilde{\mathcal{S}}$ and $X_\alpha, \alpha \in \Sigma$.
- (3) The induced morphism of sheaves $\mathcal{D}^e \rightarrow \mathcal{E}nd(\tilde{\mathcal{S}})$ is injective.

Proof. All results follow similarly as the corresponding facts in [CZZ12]. More precisely, (1) follows from Proposition 7.7, (2) follows from Lemma 5.8 and (3) follows from Theorem 7.10 of [CZZ12]. \square

4.5. Geometric meaning of the elliptic Demazure operators. In this subsection we are back to the assumption that R is a \mathbb{Q} -algebra. Let G be a connected, simply connected compact Lie group, T be a maximal torus. Note that W naturally acts on $\mathbb{X}^*(T) = \Lambda$, hence acts on \mathfrak{A}_T . The quotient \mathfrak{A}_T/W is the moduli scheme of fiber-wise topologically trivial stable principal G^{alg} -bundles over E^\vee . Let $\pi : \mathfrak{A}_T \rightarrow \mathfrak{A}_T/W$ be the natural projection.

Lemma 4.13. *Let P be a connected closed Levi subgroup of G such that $T < P$. Let W_P be the Weyl group of P . Then, there are isomorphisms $\mathfrak{A}_P \cong \mathfrak{A}_G^{G/P}$, and $\mathfrak{A}_P \cong \mathfrak{A}_T/W_P$, making following diagram is commutative.*

$$\begin{array}{ccc} \mathfrak{A}_T & \longrightarrow & \mathfrak{A}_P \\ \downarrow & \swarrow & \\ \mathfrak{A}_G & & \end{array}$$

All the maps can be identified with quotient maps by the Weyl group action.

Proof. This follow directly from Assumption 3.4 (3). \square

For any simple root α , let P_α be the corresponding minimal parabolic group. Let $p : G/T \rightarrow G/P_\alpha$ be the natural projection, which is a \mathbb{P}^1 -bundle. It induces a morphism $p^\sharp : \mathcal{E}_G^0(G/P_\alpha) \rightarrow \mathcal{E}_G^0(G/T)$, and after taking spectra, we have $p_\sharp : \mathfrak{A}_G^{G/T} \rightarrow \mathfrak{A}_G^{G/P_\alpha}$. It induces a morphism $p^\sharp : p_\sharp^* \mathcal{O}_{\mathfrak{A}_G^{G/P_\alpha}} \rightarrow \mathcal{O}_{\mathfrak{A}_G^{G/T}}$, and from Section 3.4, we have the pushforward $p_\sharp : \Theta(Tp) \rightarrow p_\sharp^* \mathcal{O}_{\mathfrak{A}_G^{G/P_\alpha}} = p_\sharp^* \mathcal{E}_G^0(G/P_\alpha)$.

Proposition 4.14. *The following diagram commutes*

$$\begin{array}{ccc} \Theta(Tp) & \xrightarrow{p^\sharp \circ p_\sharp} & \mathcal{O}_{\mathfrak{A}_G^{G/B}} \\ \downarrow \cong & & \parallel \\ \mathcal{L}_\alpha & \xrightarrow{X_\alpha} & \mathcal{O}_{\mathfrak{A}_G^{G/B}}. \end{array}$$

Proof. Without loss of generality, we assume $G = \text{PSU}_2 \subset G^\mathbb{C} = \text{PGL}_2(\mathbb{C})$ acts on \mathbb{A}^2 in the usual way, and $p : \mathbb{P}(\mathbb{A}^2) \rightarrow \text{pt}$ is the natural projection. The diagram

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{p_2} & \mathbb{P}^1 \\ p_1 \downarrow & & \downarrow p \\ \mathbb{P}^1 & \xrightarrow{p} & \text{pt} \end{array}$$

is a transversal Cartesian diagram. By base change, $p^\# \circ p_\# : \Theta(Tp) \rightarrow \mathcal{O}_{\mathfrak{A}_G^{\mathbb{P}^1}}$ is naturally equivalent to the map $p_{1\#}p_2^\# : \Theta(Tp) \rightarrow \mathcal{O}_{\mathfrak{A}_G^{\mathbb{P}^1}}$. The latter is calculated in Lemma 3.19. Therefore, we are done. \square

5. THE ELLIPTIC AFFINE HECKE ALGEBRA: THE CONVOLUTION CONSTRUCTION

In this section, we prove the isomorphism between the elliptic affine Hecke algebra and the equivariant elliptic cohomology of the Steinberg variety. Throughout this section, G^{alg} is a connected, simply-connected compact Lie group, with maximal torus T . Let $B \subseteq G^{\mathbb{C}}$ be the Borel. Let $\mathcal{B} = G^{\mathbb{C}}/B \cong G/T$ as variety over \mathbb{C} .

5.1. Convolution construction of the Demazure-Lusztig operator. Recall that in $\mathcal{B} \times \mathcal{B}$, the orbits of the diagonal $G^{\mathbb{C}}$ -action are in natural one-to-one correspondence with elements of the Weyl group W . Let Y_α be the orbit corresponding to the simple root $\alpha \in \Phi$. Its closure \overline{Y}_α is the union of Y_α and \mathcal{B}_Δ , the latter being the diagonal. Let $Z_\alpha := T_{\overline{Y}_\alpha}^*(\mathcal{B} \times \mathcal{B})$ be the conormal bundle, considered as a closed subvariety of $\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$, and denote $p_\alpha : Z_\alpha \rightarrow \overline{Y}_\alpha$. Note that Z_α is smooth, and the second projection $\text{pr}_2 : \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}}$ is proper when restricted to Z_α . Without causing confusion, we will denote the restriction of pr_i to Z_α still by pr_i for $i = 1, 2$. Via convolution and the embedding $Z_\alpha \hookrightarrow Z$, a local section η of $\Xi_{G \times S^1}(Z_\alpha)$ defines local a section of $\mathcal{E}\text{nd}_{\mathfrak{A}_{G \times S^1}}(\mathcal{E}_{G \times S^1}^0(\tilde{\mathcal{N}}))$, which will be denoted by $\eta *_Z$.

Let $p_2 : \overline{Y}_\alpha \rightarrow \mathcal{B}$ be the second projection $\mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ restricted to \overline{Y}_α . It is a fiber bundle with each fiber isomorphic to \mathbb{P}^1 . Let $\Omega_{p_2}^1$ be the relative cotangent bundle p_2 , which is a line bundle on \overline{Y}_α . Define $\mathcal{J}_\alpha := p_\alpha^* \Omega_{p_2}^1$ on Z_α .

Let f be a rational section of Ω_E . Define

$$(5) \quad J_\alpha^f := \frac{e(\mathcal{J}_\alpha)}{e(\mathcal{J}_\alpha \otimes k_q)} \cdot \left(1 - \frac{c_1^f(k_q)}{c_1^f(\mathcal{J}_\alpha)} \right),$$

which is a rational section of $\Xi_{G \times S^1}(Z_\alpha) \cong \Theta(\text{pr}_1^* T^* \mathbb{P}^1)^\vee \otimes \Theta(\mathcal{J}_\alpha)^\vee$.¹

For any local section $\sigma(x)$ of \mathcal{S} , we have

$$(6) \quad (T_{-\alpha}^f - 1) \cdot \sigma(x) = \left(\frac{f(\chi_\gamma)}{f(\chi_\alpha)} - 1 \right) (s_\alpha \sigma(x) - \sigma(x)) = \vartheta(\chi_\alpha) \left(1 - \frac{f(\chi_\gamma)}{f(\chi_\alpha)} \right) \left(\frac{s_\alpha \sigma(x)}{\vartheta(\chi_{-\alpha})} + \frac{\sigma(x)}{\vartheta(\chi_\alpha)} \right).$$

Theorem 5.1. *With notations as above,*

$$J_\alpha^f *_Z = (T_{-\alpha}^f - 1)$$

as rational sections of $\mathcal{E}\text{nd}_{\mathfrak{A}_{G \times S^1}}(\mathcal{E}_{G \times S^1}^0(\tilde{\mathcal{N}})) \cong \mathcal{E}\text{nd}_{\mathfrak{A}_{G \times S^1}} \mathfrak{A}/W(\mathcal{S})$

The proof of this theorem is similar to that of Theorem 6.3 in [ZZ14]. However, there are differences due to the phenomenon of Thom bundles in equivariant elliptic cohomology. Therefore, we include the proof for convenience of the readers.

We need the following lemma to simplify our calculation.

Lemma 5.2. [CG97, Lemma 5.4.27] *Let $j : Z_\alpha \rightarrow \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$ be the natural embedding. Let $b_2 : \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{N}} \times \mathcal{B}$ be the identity on the first factor and the bundle projection on the second factor. Let $i : \mathcal{B} \times \mathcal{B} \rightarrow \tilde{\mathcal{N}} \times \mathcal{B}$ be the zero section.*

¹Note that although Z_α 's are conormal bundles to orbits in \mathcal{B} , Proposition 2.8.6 in [GKV95] does not apply, since the S^1 -factor in $G \times S^1$ acts non-trivially on $T^* \mathcal{B}$ but trivially on \mathcal{B} .

Then, $i \circ b_2 \circ j$ is a well-defined injective map of sheaves of algebras $\pi_{Z_\alpha*} \Xi_{G \times S^1}(Z_\alpha) \rightarrow \pi_{\mathcal{B} \times \mathcal{B}*} \Xi_{G \times S^1}(\mathcal{B} \times \mathcal{B})$. Moreover, the following diagram commutes

$$\begin{array}{ccc} \pi_{Z_\alpha*} \Xi_{G \times S^1}(Z_\alpha) & \xrightarrow{*Z} & \mathcal{E}nd_{\mathfrak{A}_{G \times S^1}}(\mathcal{S}) \\ \downarrow i^* \circ b_{2*} \circ j_* & & \downarrow \cong \\ \pi_{\mathcal{B} \times \mathcal{B}*} \Xi_{G \times S^1}(\mathcal{B} \times \mathcal{B}) & \xrightarrow{*B} & \mathcal{E}nd(\mathcal{E}_{G \times S^1}^0(\mathcal{B})). \end{array}$$

5.2. Rank-1 case. Now we assume G has rank 1. The only simple root is denoted by α . In this case, $\mathcal{B} \cong \mathbb{P}^1$, and $Z_\alpha = \overline{Y_\alpha} = \mathbb{P}^1 \times \mathbb{P}^1$. We identify \mathbb{P}^1 with $\mathbb{P}(\mathbb{A}^2)$. Let $T = S^1$ acts on \mathbb{A}^2 with weights $-\alpha/2$ and $\alpha/2$. For any character λ , the line bundle \mathcal{L}_λ is isomorphic to $\mathcal{O}(\langle \lambda, \alpha^\vee \rangle)$ on \mathbb{P}^1 , and $\Omega_{\mathbb{P}^1}^1 = \mathcal{O}(-2)$. The line bundle $\mathcal{J}_\alpha = \Omega_{\mathbb{P}^2}^1$ on Z_α is identified with $\mathcal{O}(-2) \boxtimes \mathcal{O}_{\mathbb{P}^1}$. The map $i^\sharp \circ p_{2\sharp} \circ j : \pi_{Z_\alpha*} \Xi_{G \times S^1}(Z_\alpha) \rightarrow \pi_{\mathcal{B} \times \mathcal{B}*} \Xi_{G \times S^1}(\mathcal{B} \times \mathcal{B})$, according to the Thom isomorphism Theorem, is multiplication by $e(\mathcal{J}_\alpha \otimes k_q)$. Consequently, we have

$$J_\alpha^f = \frac{e(\mathcal{O}(-2) \boxtimes \mathcal{O}_{\mathbb{P}^1})}{e(\mathcal{O}(-2) \boxtimes \mathcal{O}_{\mathbb{P}^1} \otimes k_q)} \left(1 - \frac{c_1^f(k_q)}{c_1^f(\mathcal{O}(-2) \boxtimes \mathcal{O}_{\mathbb{P}^1})} \right)$$

as a rational section of $\pi_{Z_\alpha*} \Xi_{G \times S^1}(Z_\alpha)$.

Let $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the i -th projection for $i = 1, 2$. The Euler class $e(\mathcal{J}_\alpha) = \vartheta(\chi_\alpha)$ is a global section of $\pi_{\mathbb{P}^1 \times \mathbb{P}^1*} \Xi_{G \times S^1}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \pi_{\mathbb{P}^1 \times \mathbb{P}^1*} \Theta(\mathcal{J}_\alpha)^\vee$. Hence, we have the operator $e(\mathcal{J}_\alpha) *_{\mathcal{B}} \in \text{End}_{\mathfrak{A}_{G \times S^1}}(\mathcal{E}_{G \times S^1}^0(\mathbb{P}^1))$.

Lemma 5.3. *For any local section σ of $\mathcal{E}_{G \times S^1}^0(\mathbb{P}^1)$, we have*

$$e(\mathcal{J}_\alpha) *_{\mathcal{B}} \sigma = \sigma - s_\alpha \sigma.$$

Proof. We have

$$\begin{aligned} p_{1\sharp} \left(e(\mathcal{J}_\alpha) p_{2\sharp}(\sigma) \right) &= p_{1\sharp} \left(p_{1\sharp}^\sharp(e\mathcal{O}(2)) \cdot p_{2\sharp}^\sharp(\sigma) \right) \\ &= e(\mathcal{O}(2)) \cdot p_{1\sharp} \left(p_{2\sharp}^\sharp(\sigma) \right) \\ &= \vartheta(\chi_\alpha) \cdot p_{1\sharp} \left(p_{2\sharp}^\sharp(\sigma) \right). \end{aligned}$$

By Lemma 3.19, $\vartheta(\chi_\alpha) \cdot p_{1\sharp} \left(p_{2\sharp}^\sharp(\sigma) \right) = \sigma - s_\alpha \sigma$. Therefore, we are done. \square

By the projection formula and Lemma 5.3, we have the following more general formula.

Lemma 5.4. *Let σ_1 be a local section of $\mathcal{O}_{E \times E} \cong \mathcal{E}_{G \times S^1}^0(\mathbb{P}^1)$. Then any local section σ_1 of $\mathcal{E}_{G \times S^1}^0(\mathbb{P}^1)$ is sent by $e(\mathcal{J}_\alpha) *_{\mathcal{B}}$ to the following operator on $\mathcal{E}_{G \times S^1}^0(\mathbb{P}^1)$, sending any local section σ_2 to*

$$\left(e(\mathcal{J}_\alpha) \cdot p_{1\sharp}^\sharp(\sigma_1) \right) *_{\mathcal{B}} \sigma_2 = \sigma_1 \cdot (\sigma_2 - s_\alpha \sigma_2).$$

Proposition 5.5. *The effect of the operator $J_\alpha^f *_{\mathcal{B}}$ on any local section σ of $\mathcal{E}_{G \times S^1}^0(\mathbb{P}^1)$ coincides with that of $T_{-\alpha}^f - 1$ under the identification $\mathcal{S} \cong \mathcal{O}_{E \times E} \cong \mathcal{E}_{G \times S^1}^0(\mathbb{P}^1)$.*

Proof. By definition and Lemma 5.2, $J_\alpha^f *_Z \sigma = p_{1\sharp} \left[\left(p_2^\sharp(\sigma) \right) \cdot e((\Omega_{\mathbb{P}^1}^1 \otimes k_q^\vee) \boxtimes \mathcal{O}) \cdot J_\alpha^f \right] \in \mathcal{E}_{G \times S^1}^0(\mathcal{B})$. Recall that $G = \mathrm{SU}_2$, by Lemma 5.4, we have

$$\begin{aligned} p_{1\sharp} \left(\left(p_2^\sharp(\sigma) \right) \cdot e((\Omega_{\mathbb{P}^1}^1 \otimes k_q) \boxtimes \mathcal{O}) \cdot J_\alpha^f \right) &= p_{1\sharp} \left(p_2^\sharp(\sigma) \cdot e(\mathcal{J}_\alpha) \cdot p_1^\sharp \left(1 - \frac{c_1^f(k_q)}{c_1^f(\mathcal{J}_\alpha)} \right) \right) \\ &= \left(1 - \frac{f(\chi_\gamma)}{f(\chi_\alpha)} \right) (-s_\alpha \sigma + \sigma) \\ &= \left(\frac{f(\chi_\gamma)}{f(\chi_\alpha)} - 1 \right) (s_\alpha \sigma - \sigma). \end{aligned}$$

Comparing with (6) we know that the effect of J_α^f on any local section σ coincides with that of $T_{-\alpha}^f - 1$, so the conclusion follows. \square

5.3. Convolution algebra of the Steinberg variety. The goal of this subsection is to prove the following.

Theorem 5.6. *The assignment sending the rational section $J_\alpha^{f,E}$ of \mathcal{H} to the rational section J_α^f of $\Xi_{G \times S^1}(Z)$ for any simple root α and any rational function f of E extends to an isomorphism $\Upsilon : \mathcal{H} \cong \pi_{Z*} \Xi_{G \times S^1}(Z)$ of sheaves of algebras on $\mathfrak{A}_{G \times S^1}$.*

The proof goes essentially the same way as its classical counterpart in [CG97]. Nevertheless, the fact that \mathcal{H} has very few global sections makes differences. We outline the proof in the rest of this section, with emphasize on the parts that are different. For simplicity, in the remaining part of this section, we will omit the subscript $G \times S^1$ in $\Xi_{G \times S^1}(Z)$.

For any $w \in W$, let $\mathcal{Y}_w \subseteq \mathcal{B} \times \mathcal{B}$ be the orbit corresponding to $w \in W$, and denote $Z_w = T_{\mathcal{Y}_w}^*(\mathcal{B} \times \mathcal{B})$. Let $Z_{\leq w} = \sqcup_{v \leq w} Z_v$ be the closed subvariety of Z , and $i_w, i_{\leq w}$ and $i_{< w}$ be the natural embeddings. Using long exact sequence of cohomology and the vanishing of odd degree cohomology, we see that the induced map $i_{\leq w*}$ on cohomology is injective. Similarly, we have short exact sequences $0 \rightarrow \pi_{Z_{< w}*} \Xi(Z_{< w}) \rightarrow \pi_{Z_{\leq w}*} \Xi(Z_{\leq w}) \rightarrow \pi_{Z_w*} \Xi(Z_w) \rightarrow 0$, and $\Xi(Z_w)$ is a line bundle on $\mathfrak{A}_{G \times S^1}^{Z_w}$.

Lemma 5.7. *The sheaf $\pi_{Z*} \Xi(Z)$ is locally free, and the action of $\pi_{Z*} \Xi(Z)$ on $\mathcal{E}_{G \times S^1}^0(\mathcal{B})$ is faithful, i.e., the morphism $\pi_{Z*} \Xi(Z) \rightarrow \mathcal{E}nd_{\mathfrak{A}}(\mathcal{E}_{G \times S^1}^0(\mathcal{B}))$ is injective.*

Proof. Exactly the same way as [GKV95, Lemma 4.6.1]. \square

By definition of \mathcal{H} , the natural morphism $\mathcal{H} \rightarrow \mathcal{E}nd_{\mathfrak{A}/W}(\mathcal{S})$ is injective. Therefore, we get the following consequence of Lemma 5.7.

Corollary 5.8. *The assignment in Theorem 5.6 extends to a well-defined morphism $\Upsilon : \mathcal{H} \rightarrow \pi_{Z*} \Xi(Z)$ of sheaves of rings, which is injective.*

Now we finish the proof of Theorem 5.6.

Proof of Theorem 5.6. Clearly Υ is filtration preserving. It induced a morphism $\mathrm{gr}_w \Upsilon$ on associated graded piece $\mathcal{H}_w \rightarrow \pi_{Z_w*} \Xi(Z_w)$, which is a morphism of rank 1 locally free sheaves of module over \mathcal{S} .

Let $I_w = (i_1, \dots, i_r)$ be a reduced sequence of w . For any $p \in \mathfrak{A}$, we fix a collection of rational functions $\{f_{\alpha_{i_j}, p} \mid j = 1, \dots, r\}$, and an open neighborhood U of p as in § 4.2. For each simple root α , we say a local section s of $\Xi(Z_\alpha)$ is invertible if there is a local section s' of $\Xi(Z_\alpha)^\vee$ on the same

open set such that $s \otimes s'$ is $1 \in \mathcal{S}$. In particular, on U the element $\frac{e(\mathcal{J}_\alpha)}{e(\mathcal{J}_\alpha \otimes k_q)} \cdot \left(1 - \frac{c_1^{f_{\alpha_i, p}(k_q)}}{c_1^{f_{\alpha_i, p}(\mathcal{J}_\alpha)}\right)$ is invertible as a section of $\Xi(Z_{\alpha_i})$.

Following the proof of [CG97, Proposition 7.6.12.(2)], we define $\text{pr}_{j,j+1} : (T^*\mathcal{B})^{r+1} \rightarrow T^*(\mathcal{B} \times \mathcal{B}) \cong T^*(\mathcal{B}) \times T^*(\mathcal{B})$ to be the projection to the $(j, j+1)$ factor, and define $\mathcal{Z}_{i_j} = \text{pr}_{j,j+1}^{-1}(T_{Y_{s_{i_j}}}^*(\mathcal{B} \times \mathcal{B}))$, for $j = 1, \dots, r-1$. The projection

$$\mathcal{Z}_{i_1} \times_{\tilde{N}} \mathcal{Z}_{i_2} \times_{\tilde{N}} \cdots \times_{\tilde{N}} \mathcal{Z}_{i_r} \rightarrow Z_w$$

restricts to an isomorphism

$$\mathcal{Z}_{i_1} \cap \mathcal{Z}_{i_2} \cap \cdots \cap \mathcal{Z}_{i_r} \cong T_{Y_w}^*(\mathcal{B} \times \mathcal{B}).$$

Consequently, the section $J_{I_w} = J_{\alpha_{i_1}}^{f_{\alpha_{i_1}, p}} \cdots J_{\alpha_{i_r}}^{f_{\alpha_{i_r}, p}}$ is invertible in

$$\Xi(\mathcal{Z}_{i_1} \cap \mathcal{Z}_{i_2} \cap \cdots \cap \mathcal{Z}_{i_r})|_U \cong \Xi(Z_w)|_U.$$

By the same argument as [CG97, Theorem 7.6.12.], when restricted to Z_w , the convolution of J_{I_w} is equal to $\text{gr}_w \Upsilon(T_{I_w})$ (where T_{I_w} is as in Lemma 4.6). Hence, the restriction of Υ is a morphism $\mathcal{H}_w|_U \rightarrow \Xi(Z_w)|_U$ which sends the generator T_{I_w} of the source to the generator J_{I_w} of the target. So $\text{gr}_w \Upsilon$ is an isomorphism of sheaves of modules over \mathcal{S} . By the same argument as [CG97, Proposition 2.3.20.(ii).], the fact that $\text{gr} \Upsilon$ is an isomorphism implies that Υ is an isomorphism. \square

6. GEOMETRIC CONSTRUCTION OF REPRESENTATIONS AT NON-TORSION POINTS

In this section, we assume E is an elliptic curve over \mathbb{C} (although all the results in this section are true if more generally the base field is characteristic zero). Again, G is a connected, simply-connected compact Lie group.

As \mathcal{H} is a coherent sheaf of algebras on the Noetherian scheme \mathfrak{A}/W , any irreducible representation of \mathcal{H} is supported on a closed point of \mathfrak{A}/W . We generalize Kazhdan–Lusztig’s classification of irreducible representations of affine Hecke algebra [KL87] to the case of elliptic affine Hecke algebra at closed points of \mathfrak{A}/W whose γ -coordinate is non-torsion.

Definition 6.1. A representation of \mathcal{H} is a coherent sheaf on \mathfrak{A}/W , which is a sheaf of modules over \mathcal{H} .

6.1. Reminder on the decomposition theorem. We recall briefly basic facts about representations of convolution algebras, following [CG97, § 8.6].

Let $f : M \rightarrow N$ be a projective morphism of quasi-projective complex varieties, with M smooth. Let $D_c^b(N)$ be the derived category of constructible sheaves on N . Let $X = M \times_N M$. Then $H_*(X; \mathbb{C})$ is endowed with a convolution product, and $H_*(X; \mathbb{C}) \cong \text{Ext}_{D_c^b(N)}^*(f_* \mathbb{C}_M)$ as associative algebras. Applying the decomposition theorem [BBD82] to $f_* \mathbb{C}_M$, we get

$$f_* \mathbb{C}_M \cong \bigoplus_{\phi, k} L_{\phi, k} \otimes P_{\phi}[k],$$

where k runs through \mathbb{Z} and ϕ runs through the set of isomorphism classes of simple perverse sheaves. Consequently, $\{L_{\phi, 0} \mid L_{\phi, 0} \neq 0\}$ is a complete set of pair-wise non-isomorphic simple modules over $H_*(X; \mathbb{C})$ (see, e.g., [CG97, Theorem 8.6.12]).

If furthermore f is a G -equivariant map between G -varieties, such that N has only finitely many orbits, then we can label the set of simple perverse sheaves ϕ in the decomposition above by pairs (\mathbb{O}, χ) , where \mathbb{O} is an orbit in N , and χ is an equivariant local system on \mathbb{O} (see, e.g., [CG97, Theorem 8.4.12]). Choosing a base point $x_{\mathbb{O}}$ for each orbit \mathbb{O} , and writing its isotropy subgroup

as $G_{x_\mathbb{O}}$, then the local system χ is identified with an irreducible representation of the component group $G(x_\mathbb{O})/G(x_\mathbb{O})^0$ (see, e.g., [CG97, 8.4.13.(ii)]).

In the equivariant set-up, we define the standard $H^*(X, \mathbb{C})$ -modules as follows. For each \mathbb{O} , let $x \in \mathbb{O}$ and let $H_*(M_x)$ be $H^*(i_x^! f_* \mathbb{C}_M)$ with the $H^*(X)$ -module structure, where $i_x : \{x\} \hookrightarrow N$ is the embedding. For each $\phi = (\mathbb{O}_\phi, \chi_\phi)$, let $H_*(M_x)_\phi$ be the component in $H_*(M_x)$ that transforms as ϕ under the component group action. This is a submodule of $H_*(M_x)$. We have

$$H_*(M_x) \cong \bigoplus_{\phi} L_{\phi} \otimes H^*(i_x^! IC_{\phi}),$$

which does not depend on the choice of $x_\mathbb{O}$. Then, the standard module $H_*(M_x)_\chi$ has a simple top, which is isomorphic to the simple module $L_{\phi,0}$. For any two parameters $\psi = (\mathbb{O}_\psi, \chi_\psi)$ and $\phi = (\mathbb{O}_\phi, \chi_\phi)$, the multiplicity of the simple object $L_{\phi,0}$ in the standard module $H_*(M_x)_\psi$ is given by

$$[H_*(M_x)_\psi : L_\phi] = \sum_k \dim H^k(i_x^! P_\phi)_\psi$$

(see, e.g., [CG97, Theorem 8.6.23]). Here the simple perverse sheaf P_ϕ is the intersection cohomology sheaf IC_ϕ associated to the local system χ_ϕ on \mathbb{O}_ϕ .

6.2. Pontrjagin duality. We recall some well-known facts about the moduli space \mathfrak{A}_G . In a simply connected compact Lie group G , for any pair of commuting elements $s_1, s_2 \in G$, there is a maximal torus $T < G$ and some $g \in G$ such that $g \cdot s_1 \cdot g^{-1}, g \cdot s_2 \cdot g^{-1} \in T$. Fix a maximal torus $T < G$, for any two pairs of elements $(g_1, g_2), (h_1, h_2) \in T^2$, if there is $g \in G$ such that $g \cdot g_1 \cdot g^{-1} = h_1$ and $g \cdot g_2 \cdot g^{-1} = h_2$, then this g can be chosen from the normalizer of T in G . It is well-known that a closed point in \mathfrak{A}_G corresponds to an ordered pair of semi-simple elements in T , up to simultaneous conjugation. Equivalently, $\mathfrak{A}_G \cong T \times T/W$. When $G = T$, this yields $\mathfrak{A}_T \cong T^2$.

This isomorphism can be made more explicit, after fixing an isomorphism $E \cong S^1 \times S^1$. By definition of \mathfrak{A}_T , any closed point $a \in \mathfrak{A}_T$ defines a homomorphism of abelian groups $a : \mathbb{X}^*(T) \rightarrow E$, hence defines an element in $\text{Hom}(\mathbb{X}^*(T), S^1 \times S^1) \cong T^2$. This assignment sending $a \in \mathfrak{A}$ to $a \in T^2$ is denoted by

$$DD : \mathfrak{A}_T \rightarrow T^2.$$

Lemma 6.2. *Let $a \in \mathfrak{A}_T$ be any closed point, and let $DD(a) = (s_1, s_2) \in T^2$. Recall that $T(a) = \bigcap_{a \in \mathfrak{A}_{T'}} T'$. Then $T(a) \subseteq T$ is the minimal closed subgroup of T generated by s_1 and s_2 .*

Proof. Closed subgroups of T form a lattice under inclusions, and so do subgroups of $\mathbb{X}^*(T)$. There is an order-reversing one-to-one correspondence between these two lattices, sending any $T' < T$ to the kernel of the quotient $\mathbb{X}^*(T) \twoheadrightarrow \mathbb{X}^*(T')$.

Let T' be a closed subgroup of T . Then $a \in \mathfrak{A}_{T'} \subseteq \mathfrak{A}_T$ if and only if the morphism $a : \mathbb{X}^*(T) \rightarrow E$ factors through $\mathbb{X}^*(T) \twoheadrightarrow \mathbb{X}^*(T')$. In turn, this happens if and only if $DD(a)$ is contained in the subgroup $T' \times T' \subseteq T \times T$. Therefore, the smallest closed subgroup $T' < T$ with the property that $a \in \mathfrak{A}_{T'} \subseteq \mathfrak{A}_T$ is also the smallest closed subgroup of $T' < T$ with the property that $DD(a) \in T' \times T'$. The later is the closed subgroup of T generated by s_1 and s_2 . \square

Let T^{alg} be the corresponding algebraic torus containing T as a maximal compact subgroup. Clearly, $\text{Hom}(\mathbb{X}^*(T), \mathbb{C}^*) \cong T^{\text{alg}}$, and $\text{Hom}(\mathbb{X}^*(T), S^1) \cong T$. Intersection with T defines an inclusion-preserving one-to-one correspondence between algebraic subgroups of T^{alg} and closed subgroups of T . In particular, for any subset $Z \subseteq T$, the smallest algebraic subgroup of T^{alg} containing Z corresponds to the closed subgroup of T generated by Z . Composing $DD : \mathfrak{A}_T \rightarrow T^2$

with the natural embedding and the embedding $T^2 \subseteq (T^{\text{alg}})^2$, we get a map $\mathfrak{A}_T \rightarrow (T^{\text{alg}})^2$, which, without causing confusion, is also denoted by DD .

6.3. A non-vanishing theorem. In this subsection, we fix a quintuple (s_1, q_1, s_2, q_2, x) , where $s_i \in T$, $q_i \in \mathbb{C}^*$, and $x \in \mathcal{N}$ such that $s_i x s_i^{-1} = q_i x$ for $i = 1, 2$. We assume that q_1 and q_2 are not simultaneously roots of unity. Without loss of generality, we assume q_1 has infinite order. Let $u = e^x \in G^{\text{alg}}$. In this subsection we prove the non-vanishing theorem (Proposition 6.7). The proof is similar to [KL87, § 7.1] (see also [CG97, § 8]).

Following [KL87, § 7.1], we fix (not-necessarily continuous) group homomorphisms $v_i : \mathbb{C}^* \rightarrow \mathbb{R}$ (the additive group) for $i = 1, 2$ such that $v_1(q_1) > 0$ and $v_2(q_2) \geq 0$. The existence of such v_2 follows from [CG97, Lemma 8.8.12]. Also fix $\phi : \text{SL}_2(\mathbb{C}) \rightarrow G^{\text{alg}}$ such that $\phi(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) = u$. For $i = 1, 2$, let $s_i^\phi = \phi(\begin{smallmatrix} q_i & 0 \\ 0 & q_i^{-1} \end{smallmatrix})$, and let $s'_i = (s_i^\phi)^{-1} \cdot s_i$. One can easily check that s'_i commutes with $\phi(\text{SL}_2(\mathbb{C}))$ for $i = 1, 2$, and s'_1 commutes with s'_2 . Then there is a decomposition of \mathfrak{g} into simultaneous eigenspaces of s'_1 , s'_2 , and $\phi(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix})$ for any $z \in \mathbb{C}^*$ as follows

$$(7) \quad \mathfrak{g} = \bigoplus_{\alpha_1 \in \mathbb{C}^*, \alpha_2 \in \mathbb{C}^*, j \in \mathbb{Z}} \mathfrak{g}_{\alpha_1, \alpha_2, j},$$

where

$$\mathfrak{g}_{\alpha_1, \alpha_2, j} = \{y \in \mathfrak{g} \mid s'_i \cdot y \cdot (s'_i)^{-1} = \alpha_i y \text{ for } i = 1, 2, \text{ and } \phi(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}) \cdot y \cdot \phi(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix})^{-1} = z^j y\}.$$

Define $\mathfrak{p}_i = \bigoplus_{v_i(\alpha_i) \leq 0} \mathfrak{g}_{\alpha_1, \alpha_2, j}$ for $i = 1, 2$, and define $\mathfrak{p} = \mathfrak{p}_1 \cap \mathfrak{p}_2$. Let the parabolic subgroup of G corresponding to \mathfrak{p}_i be P_i for $i = 1, 2$, and let P be that corresponding to \mathfrak{p} . Let L_i be the Levi subgroup of P_i for $i = 1, 2$, and L be that of P .

Let $G(s_1, s_2)$ (resp. $G(s_1, s_2, x)$) be the simultaneous centralizer of s_1 and s_2 (resp. s_1, s_2 and x) in G^{alg} , and let $\mathfrak{g}(s_1, s_2)$ (resp. $\mathfrak{g}(s_1, s_2, x)$) be its Lie algebra. Define $\mathfrak{g}^{s_1, s_2} = \{y \in \mathfrak{g} \mid s_i \cdot y \cdot s_i^{-1} = q_i y \text{ for } i = 1, 2\}$. We write $\mathfrak{g}(s_1, s_2) \cap \mathfrak{p}$ as $\mathfrak{p}(s_1, s_2)$, and $\mathfrak{p} \cap \mathfrak{g}^{s_1, s_2}$ as \mathfrak{p}^{s_1, s_2} .

The following lemma is essential in the proof of the non-vanishing theorem. Its proof is almost the same as that of [CG97, Lemma 8.8.22]. The key assumption is that $v_1(q_1)$ is strictly positive.

Lemma 6.3. *The following is true:*

- (1) $\mathfrak{g}(s_1, s_2, x) \subseteq \bigoplus_{\alpha_1 = q_1^{-j}, \alpha_2 = q_2^{-j}, j \geq 0} \mathfrak{g}_{\alpha_1, \alpha_2, j};$
- (2) $\mathfrak{p}(s_1, s_2) = \bigoplus_{\alpha_1 = q_1^{-j}, \alpha_2 = q_2^{-j}, j \geq 0} \mathfrak{g}_{\alpha_1, \alpha_2, j};$
- (3) $\mathfrak{p}^{s_1, s_2} = \bigoplus_{\alpha_1 = q_1^{2-j}, \alpha_2 = q_2^{2-j}, j \geq 2} \mathfrak{g}_{\alpha_1, \alpha_2, j};$
- (4) $[x, \mathfrak{p}(s)] = \mathfrak{p}^{s_1, s_2}.$

Proof. Recall that s'_i commutes with the image of ϕ for $i = 1, 2$. Hence, the decomposition (7) induces decompositions of $\mathfrak{g}(s_1, s_2, x)$, $\mathfrak{p}(s_1, s_2)$, and \mathfrak{p}^{s_1, s_2} .

For any $y \in \mathfrak{g}(s_1, s_2, x) \cap \mathfrak{g}_{\alpha_1, \alpha_2, j}$, we have

$$y = s_i \cdot y \cdot s_i^{-1} = s_i^\phi \cdot (s'_i \cdot y \cdot (s'_i)^{-1}) \cdot (s_i^\phi)^{-1} = \alpha_i q_i^j y.$$

Hence $\alpha_i = q_i^{-j}$ for $i = 1, 2$. Recall that $x \in \mathfrak{g}$ is nilpotent, and $u = e^x$ extends to the SL_2 -triple $\langle u, \phi(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}), \phi(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \rangle$. Let $G(u)$ be the centralizer of u , and $\mathfrak{g}(u)$ its Lie algebra. By [CG97, Corollary 3.7.11], the eigenvalues of $\phi(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix})$ on $\mathfrak{g}(s_1, s_2, x) \subseteq \mathfrak{g}(u)$ are all non-negative. Therefore, if $\mathfrak{g}(s_1, s_2, x) \cap \mathfrak{g}_{\alpha_1, \alpha_2, j}$ contains y , then we have $j \geq 0$.

Similarly, for $y \in \mathfrak{p}(s_1, s_2) \cap \mathfrak{g}_{\alpha_1, \alpha_2, j}$, we also have $\alpha_i = q_i^{-j}$ for $i = 1, 2$. We have $v_i(\alpha_i) = -jv_i(q_i)$ for $i = 1, 2$. By definition of \mathfrak{p} , we have $v_i(\alpha_i) \leq 0$ for $i = 1, 2$. But $v_1(q_1) < 0$, hence $j \geq 0$.

The decomposition in (3) is proved in the same way as (2).

To prove (4), we only need to show surjectivity of

$$\text{ad } x : \bigoplus_{\alpha_1=q_1^{-j}, \alpha_2=q_2^{-j}, j \geq 0} \mathfrak{g}_{\alpha_1, \alpha_2, j} \rightarrow \bigoplus_{\alpha_1=q_1^{2-j}, \alpha_2=q_2^{2-j}, j \geq 2} \mathfrak{g}_{\alpha_1, \alpha_2, j},$$

which is equivalent to surjectivity of

$$\text{ad } x : \mathfrak{g}_{q_1^{-j}, q_2^{-j}, j} \rightarrow \mathfrak{g}_{q_1^{-j}, q_2^{-j}, j+2}$$

for $j \geq 0$. Indeed, the operator $\text{ad } x$ extends to an \mathfrak{sl}_2 -action, which is the image of ϕ . According to the general theory of \mathfrak{sl}_2 -representation, $\text{ad } x$ is surjective on positive weight spaces. \square

Let $\mathcal{B}_x^{s_1, s_2}$ be the variety of Borel subgroups B such that $s_1, s_2 \in B$ and $x \in \text{Lie } B$, and let $\mathcal{B}_x^{P, s_1, s_2} = \{B \in \mathcal{B}_x^{s_1, s_2} \mid B \subseteq P\}$. Then $\mathcal{B}_x^{P, s_1, s_2} \neq \emptyset$, since any solvable subgroup of G^{alg} is contained in at least one Borel subgroup.

Let $\mathbb{O} \subseteq \mathcal{N}$ be the $G(s_1, s_2)$ -orbit of $x \in \mathcal{N}$, and let $\overline{\mathbb{O}}$ be the closure of \mathbb{O} . The following lemma follows is an analogue of [CG97, Proposition 8.8.19].

Lemma 6.4. *Let $B \in \mathcal{B}_x^{P, s_1, s_2}$ be a Borel subalgebra, and let \mathfrak{n} be the nil-radical of $\text{Lie } B$. Then*

$$G(s_1, s_2) \cdot (\mathfrak{n} \cap \mathfrak{g}^{s_1, s_2}) = \overline{\mathbb{O}}.$$

Proof. By definition, we have $x \in \mathfrak{g}^{s_1, s_2}$ and $x \in \mathfrak{n}$. Therefore, to prove this lemma, we only need to show that $G(s_1, s_2)x$ is dense in $G(s_1, s_2)\mathfrak{p}^{s_1, s_2}$.

Note that $x \in \mathfrak{p}^{s_1, s_2}$, and that \mathfrak{p}^{s_1, s_2} is stable under the action of $P(s_1, s_2) := P \cap G(s_1, s_2)$. We have a proper map

$$G(s_1, s_2) \times_{P(s_1, s_2)} \mathfrak{p}^{s_1, s_2} \twoheadrightarrow G(s_1, s_2) \cdot \mathfrak{p}^{s_1, s_2}.$$

To show that $G(s_1, s_2)x$ is dense in $G(s_1, s_2)\mathfrak{p}^{s_1, s_2}$, we only need to show that the $P(s_1, s_2)$ -adjoint orbit of x is dense in \mathfrak{p}^{s_1, s_2} . But this follows directly from Lemma 6.3 and [CG97, Lemma 1.4.12]. \square

Let $\tilde{\mathcal{N}}^{(s_1, q_1), (s_2, q_2)}$ be the subset of $\tilde{\mathcal{N}}$ consisting of elements fixed by both (s_1, q_1) and (s_2, q_2) . Here the action of $(s, q) \in G^{\text{alg}} \times \mathbb{C}^*$ on $n \in \tilde{\mathcal{N}}$ is given by $q^{-1} \cdot (s \cdot n \cdot s^{-1})$. Define $\hat{\mathbb{O}}$ to be the union of connected components of $\tilde{\mathcal{N}}^{(s_1, q_1), (s_2, q_2)}$ that intersect non-trivially with $\mathcal{B}_x^{P, s_1, s_2}$. Then $\hat{\mathbb{O}}$ is a $G(s_1, s_2)$ -stable subvariety of $\tilde{\mathcal{N}}^{(s_1, q_1), (s_2, q_2)}$, and is both open and closed.

So far we proved the following lemma, which is an analogue of [CG97, Theorem 8.8.1].

Lemma 6.5. *With notations as above, we have $\mu(\hat{\mathbb{O}}) = \overline{\mathbb{O}}$.*

Let $L(s_1, s_2, x)$ be the simultaneous centralizer of s_1, s_2, x in L . Let $C(s_1, s_2, x)$ be the component group of $G(s_1, s_2, x)$, i.e., the quotient of $G(s_1, s_2, x)$ by its connected component containing the identity. Then, $C(s_1, s_2, x)$ acts on $H^*(\mathcal{B}_x^{s_1, s_2})$ and $H^*(\widehat{\mathcal{B}_x^{s_1, s_2}})$, where $\widehat{\mathcal{B}_x^{s_1, s_2}} = \mathcal{B}_x^{s_1, s_2} \cap \hat{\mathbb{O}}$.

Lemma 6.6. *We have*

- (1) $G(s_1, s_2, x) \subseteq P$;
- (2) *the map of component groups $L(s_1, s_2, x)/L(s_1, s_2, x)^0 \rightarrow C(s_1, s_2, x)$ is surjective.*

Proof. For $i = 1, 2$, let

$$M(s_i, x) = \{(g, q) \in G \times \mathbb{C}^* \mid gs_i g^{-1} = s_i, gxg^{-1} = q^2 x\},$$

and let $M(s_1, s_2, x) = M(s_1, x) \cap M(s_2, x)$. It is shown in [KL87, Lemma 7.2(a)] (see also [CG97, Lemma 8.8.23]) that $M(x, s_i) \subseteq P_i \times \mathbb{C}^*$ for $i = 1, 2$. Hence, $M(x, s_1) \cap M(x, s_2) = M(x, s_1, s_2) \subseteq P \times \mathbb{C}^* = (P_1 \times \mathbb{C}^*) \cap (P_2 \times \mathbb{C}^*)$. In particular, $G(s_1, s_2, x) \subseteq P$.

Claim (2) follows similarly as [CG97, Lemma 8.8.25]. \square

The following non-vanishing theorem is an analogue of [CG97, Proposition 8.8.2]. For convenience of the readers, we include the proof.

Proposition 6.7. *Any simple $C(s_1, s_2, x)$ -module having non-zero multiplicity in $H^*(\mathcal{B}_x^{s_1, s_2})$ also has non-zero multiplicity in $H^*(\widehat{\mathcal{B}_x^{s_1, s_2}})$.*

Proof. Let $Z^0(L)$ be the identity component of the center of L . We have $H^*(\mathcal{B}_x^{s_1, s_2}) \cong H^*((\mathcal{B}_x^{s_1, s_2})^{Z^0(L)})$. In particular, for any simple $C(x, s_1, s_2)$ -module χ , the multiplicity $[H^*(\mathcal{B}_x^{s_1, s_2}) : \chi]$ is non-zero if and only if $[H^*((\mathcal{B}_x^{s_1, s_2})^{Z^0(L)}) : \chi]$ is non-zero.

Let \mathcal{B}^P be the variety consisting of all the Borel subgroups contained in P . By [CG97, (8.8.28)], the flag variety $\mathcal{B}(L)$ of L is isomorphic to \mathcal{B}^P . Also, by [CG97, Proposition 8.8.2], each connected component of $\mathcal{B}^{Z^0(L)}$ is L -equivariantly isomorphic to $\mathcal{B}(L)$. This in turn implies that $(\mathcal{B}_x^{s_1, s_2})^{Z^0(L)}$ is a disjoint union of pieces, each $L(s_1, s_2, x)$ -equivariantly isomorphic to $\mathcal{B}(L)_x^{s_1, s_2} \cong \mathcal{B}_x^{P, s_1, s_2}$. Therefore, $H^*((\mathcal{B}_x^{s_1, s_2})^{Z^0(L)}) \cong H^*(\mathcal{B}_x^{P, s_1, s_2})^{\oplus m}$ for some $m > 0$ as $L(s_1, s_2, x)/L(s_1, s_2, x)^0$ -modules.

By Lemma 6.6, for any simple $C(s_1, s_2, x)$ -module χ such that $[H^*((\mathcal{B}_x^{s_1, s_2})^{Z^0(L)}) : \chi] \neq 0$, we also have $[H^*(\mathcal{B}_x^{P, s_1, s_2}) : \chi] \neq 0$. Recall that by definition we have $\mathcal{B}_x^{P, s_1, s_2} \subseteq \widehat{\mathcal{B}_x^{s_1, s_2}}$, and hence $(\widehat{\mathcal{B}_x^{s_1, s_2}})^{Z^0(L)}$ contains $\mathcal{B}_x^{P, s_1, s_2}$ as a union of connected components. Therefore, we then also have $[H^*(\widehat{\mathcal{B}_x^{s_1, s_2}})^{Z^0(L)} : \chi] \neq 0$. This in turn implies that $[H^*(\widehat{\mathcal{B}_x^{s_1, s_2}}) : \chi] \neq 0$, which finishes the proof. \square

6.4. Classification of irreducible representations at non-torsion points. Let $(a, t) \in \mathfrak{A}_T \times E$ be a closed point, such that $t \in E$ is not a torsion point.

Let $DD(a, t) = ((s_1, q_1), (s_2, q_2)) \in (T \times S^1)^2$. The condition that t is non-torsion is equivalent to saying that q_1 and q_2 are not simultaneously roots of unity. Let $T(a, t) \subseteq T \times S^1$ be the closed subgroup generated by (s_1, q_1) and (s_2, q_2) . Let $x \in \mathcal{N}^{T(a, t)}$ be a nilpotent element fixed by the subgroup $T(a, t)$. Then, we have $s_i x s_i^{-1} = q_i x$ for $i = 1, 2$. Let $T(a) < T$ be as usual, and let $G(a)$ be the centralizer of $T(a)$ in G^{alg} . By Lemma 6.2, $T(a)$ is generated by s_1 and s_2 , hence $G(a) = G(s_1, s_2)$ where the latter is defined in § 6.3. Let $G(a, x) < G(a)$ be the centralizer of x in $G(a)$, then we have $G(a, x) = G(s_1, s_2, x)$. The component group $C(a, x)$ of $G(a, x)$ is the same as $C(s_1, s_2, x)$ in § 6.3.

We need some basic properties about the equivariant elliptic cohomology of the Steinberg variety Z .

Lemma 6.8. *Let $\pi : \mathfrak{A}_{T \times S^1}^Z \rightarrow \mathfrak{A}_{G \times S^1}^Z$ be the natural projection. Then*

$$\pi^* \Xi_{G \times S^1}(Z) \cong \Xi_{T \times S^1}(Z)$$

as sheaves of algebras on $\mathfrak{A}_{T \times S^1}^Z$.

The K -theory analogue of this lemma is [CG97, (6.2.(6))]. In the elliptic case, it follows from the fact that $\pi_{Z*} \Xi_{T \times S^1}(Z)$ is locally free on $\mathfrak{A}_{T \times S^1}$, together with faithfully flat descent for the map $\pi : \mathfrak{A}_{T \times S^1} \rightarrow \mathfrak{A}_{G \times S^1}$, similar to its K -theory analogue.

For any $t \in E$, let \mathcal{H}_t be the pull-back of \mathcal{H} to the subvariety $\mathfrak{A}_G \times \{t\} \subseteq \mathfrak{A}_G \times E$.

Proposition 6.9 ([CG97], (8.1.6)). *Let $(a, t) \in \mathfrak{A}_G \times E$, $\mathcal{H}_{(a,t)}$ be the pull-back of \mathcal{H} to the closed point (a, t) . Let $a' \in \mathfrak{A}_T$ be any point in the pre-image of $a \in \mathfrak{A}_G$. For simplicity, $T(a', t) < T \times S^1$ is denoted by $T(a, t)$. Then we have an isomorphism of algebras*

$$\mathcal{H}_{(a,t)} \cong H_*(Z^{T(a,t)}; \mathbb{C}).$$

Proof. Let $\iota : T(a, t) \hookrightarrow T \times S^1 \subseteq G \times S^1$ be the natural embedding, and let $\iota_{\mathfrak{A}} : \mathfrak{A}_{T(a,t)} \rightarrow \mathfrak{A}_{G \times S^1}$ be the induced map. By Lemma 6.8, we have $\pi^* \Xi_{G \times S^1}(Z) \cong \Xi_{T \times S^1}(Z)$ as vector bundles on $\mathfrak{A}_{T \times S^1}^Z$, hence $\iota_{\mathfrak{A}}^* \pi_{Z*} \Xi_{T \times S^1}(Z) \cong \pi_{Z*} \Xi_{T(a,t)}(Z)$. In particular,

$$\mathcal{H}_{(a,t)} \cong (\pi_{Z*} \Xi_{T(a,t)}(Z)) \otimes_{\mathfrak{A}_{T(a,t)}} \mathbb{C}_{(a,t)}.$$

By Theorem 3.25, we have an isomorphism $\pi_{Z*} \Xi_{T(a,t)}(Z) \otimes_{\mathfrak{A}_{T(a,t)}} \mathbb{C}_{(a,t)} \cong H_*(Z^{T(a,t)}; \mathbb{C})$. \square

Apply § 6.1 to $\mu : \tilde{\mathcal{N}}^{T(a,t)} \rightarrow \mathcal{N}^{T(a,t)}$ with the action of $G(a)$, we get the following corollary of Proposition 6.7.

Corollary 6.10. *Assume $t \in E$ is a non-torsion point. Then,*

- (1) *For $a \in \mathfrak{A}_T$, $x \in \mathcal{N}^{T(a,t)}$, and χ an irreducible representation of $C(a, x)$ that shows up non-trivially in $H^*(\mathcal{B}_x^{T(a)})$, the \mathbb{C} -vector space $H_*(\mathcal{B}_x^{T(a)})_{\chi}$ has a natural \mathcal{H}_t -module structure.*
- (2) *For each triple (a, x, χ) with a, x , and χ as in (1), the module $H_*(\mathcal{B}_x^{T(a)})_{\chi}$ has a simple top, denoted by $L_{a,x,\chi}$. Moreover, the set of irreducible representations of \mathcal{H}_t are in one-to-one correspondence with the set of triples (a, x, χ) up to conjugation;*
- (3) *Let (a, y, κ) be another triple. The multiplicity of the simple object $L_{a,x,\chi}$ in $H_*(\mathcal{B}_y^{T(a)})_{\kappa}$ is given by $\sum_k \dim H^k(i_x^! IC_{x,\chi})_{y,\kappa}$.*

Here $IC_{x,\chi}$ is the intersection cohomology sheaf on $\mathcal{N}^{T(a)}$ associated to the local system χ on the orbit $G(a)x$, and $H^k(i_y^! IC_{x,\chi})_{\kappa}$ is the $C(a, y)$ -isotypical component transforms as κ . This corollary follows directly from Proposition 6.7 by the same argument as in [CG97, § 8.8].

Remark 6.11. ² As a direct application of [Kal08, Corollary 1.10], $H^*(\mathcal{B}_x^{T(a)})$ is generated by algebraic cycles. In particular, the Euler characteristic of $\mathcal{B}_x^{T(a)}$ is the same as its dimension.

6.5. Higgs bundle interpretations. For any principle G^{alg} -bundle on E^{\vee} , let $\text{ad}(P)$ be the Lie algebra bundle $P \times_{G^{\text{alg}}} \mathfrak{g}$. For any $t \in E$, denote the line bundle $\mathcal{O}(\{0\} - \{t\})$ on E^{\vee} by \mathcal{L}_t . A \mathcal{L}_t -valued *Higgs field* is a holomorphic section³ of the vector bundle $\text{ad}(P) \otimes \mathcal{L}_t$. The pair (P, x) is called a *Higgs bundle*. A B -structures on the Higgs bundle (P, x) is a principal B -bundle P' on E^{\vee} , together with a holomorphic section x' of $\text{ad}(P') \otimes \mathcal{L}_t$, such that (P, x) is induced from (P', x') . Let $\mathbf{B}_{P,x}$ be the variety of all the B -structures on the Higgs bundle (P, x) .

Theorem 6.12. *For any $t \in E$ and any $a \in \mathfrak{A}_G$, let \mathcal{L}_t be the line bundle corresponding to t , and let P_a be the principal G^{alg} -bundle corresponding to a . Then, we have*

- (1) *A holomorphic section x of $\text{ad}(P_a) \otimes \mathcal{L}_t^{-1}$ is equivalent to $x \in \mathfrak{g}^{T(a,t)}$;*
- (2) *There is an isomorphism of algebraic varieties $\mathbf{B}_{P_a,x} \cong \mathcal{B}_x^{T(a)}$;*
- (3) *There is an isomorphism of algebraic groups $\text{Aut}(P, x) \cong G(a, x)$, which intertwines the action of $\text{Aut}(P, x)$ on $\mathbf{B}_{P_a,x}$ and the action of $G(a, x)$ on $\mathcal{B}_x^{T(a)}$.*

²This remark was communicated to the authors by Sasha Braverman in the Park City Mathematics Institute 2015.

³For a \mathfrak{g} -bundle, we use the terminology *holomorphic section* instead of *regular section* to avoid confusion with regularity of elements in \mathfrak{g} .

We will prove this theorem in the end of this section. First, let us mention an application. Let $Higgs_t^{nil}(E^\vee)$ be the set of isomorphism classes of triples (P, x, χ) , where P is a semi-simple semi-stable topologically trivial principle G^{alg} -bundle on E^\vee , x is a nilpotent holomorphic section of $\text{ad}(P) \otimes \mathcal{L}_t^{-1}$, and χ is an irreducible representation of $\text{Aut}(P, x)/\text{Aut}^0(P, x)$ on $H^*(\mathbf{B}_{P,x})$. Here, a nilpotent section of $\text{ad}(P) \otimes \mathcal{L}_t$ is a section such that the image at each fiber is an nilpotent element of \mathfrak{g} .

- Remark 6.13.** (1) For $t = 0$, the set $Higgs_{t=0}^{nil}(E^\vee)$ is shown to parametrize irreducible representations of the quantum torus in [BEG03].
- (2) Moreover, for non-torsion $t \in E$, it was conjectured in *loc. cit.* that the set $Higgs_t^{nil}(E^\vee)$ parametrizes irreducible representations of the double affine Hecke algebra of Cherednik (DAHA).
- (3) However, in [Vas05] a classification of those representations has been achieved, which shows more those representations than the ones predicted by this conjecture.

Therefore, Remark 6.13 gives raise to two natural problems.

- Problem 6.14.** (1) Find a family of algebras, parametrized by $t \in E$, whose irreducible representations are parametrized by $Higgs_t^{nil}(E^\vee)$ for non-torsion point $t \in E$.
- (2) Find a property P , such that the irreducible integrable representations of DAHA having property P are parametrized by $Higgs_t^{nil}(E^\vee)$.

The following is a direct corollary of Theorem 6.12.

Corollary 6.15. *Let $t \in E$ be a non-torsion point, then the irreducible objects in $\mathcal{H}_t\text{-mod}$ are in one-to-one correspondence with $Higgs^{nil}(E^\vee)$.*

Corollary 6.15 answers Problem 6.14(1). In a later publication of the first named author in collaboration with Valerio Toledano Laredo, we will address Problem 6.14(2). The property P is expected to be *non-torsion* when considered as a module over one of the subalgebras isomorphic to the affine Hecke algebra.

The proof of Theorem 6.12 will occupy the rest of this section. As an intermediate step, we interpret $Higgs_t^{nil}(E^\vee)$ in terms of principal bundles with flat connections.

Let $\pi_1 := \pi_1(E, 0)$ be the fundamental group of E with base point $0 \in E$, and let $\tilde{E} \rightarrow E$ be the universal cover. Denote the two generators of π_1 by γ_1 and γ_2 . For $a \in \mathfrak{A}_G$, let $(s_1, s_2) \in (T^{\text{alg}})^2$ be $DD(a)$ as in § 6.2, and for $t \in E$ we associate $(q_1, q_2) \in (S^1)^2 \subseteq \mathbb{G}_m^2$. Note that by construction s_i is unitary, in the sense that s_i lies in the maximal compact subgroup $T \subset T^{\text{alg}}$. Let $\rho : \pi_1 \rightarrow T \subset G^{\text{alg}}$ be the group homomorphism sending γ_i to s_i , and let $\eta : \pi_1 \rightarrow S^1 \subset \mathbb{G}_m$ be the group homomorphism sending γ_i to q_i , for $i = 1, 2$.

The group π_1 acts on \tilde{E} by deck transform, and acts on \mathfrak{g} via $\rho : \pi_1 \rightarrow T \subset G^{\text{alg}}$ and the adjoint action of G on \mathfrak{g} . Define \tilde{a}_ρ to be the local \mathfrak{g} -system on \tilde{E} , endowed with the diagonal π_1 -action. The quotient \tilde{a}_ρ/π_1 is a local \mathfrak{g} -system on E , denoted by a_ρ . Note that as a vector bundle on E , a_ρ is the adjoint bundle $\text{ad}(a)$. The construction above endows $\text{ad}(a)$ with a unitary flat connection.

Similarly, we have the rank-1 local system \tilde{t}_η on \tilde{E} , whose descent to E is denoted by t_η . This local system t_η can be considered as a unitary flat line bundle, whose underlying line bundle is \mathcal{L}_t .

Proposition 6.16. *The following holds.*

- (1) *A flat section x of $a_\rho \otimes_{\mathbb{C}} t_\eta^{-1}$ is equivalent to an element $x \in \mathfrak{g}^{T(a,t)}$;*
- (2) *There is an isomorphism of algebraic varieties between $\mathcal{B}_x^{T(a)}$ and $\mathbf{B}_{a_\rho, x}$, the latter being the variety of B -structures on the pair (a_ρ, x) ;*

- (3) *There is an isomorphism of algebraic groups $\text{Aut}(a_\rho, x) \cong G(a, x)$, which intertwines the action of $\text{Aut}(a_\rho, x)$ on $\mathbf{B}_{a_\rho, x}$ and the action of $G(a, x)$ on $\mathcal{B}_x^{T(a)}$.*

Proof. (1). Note that $\tilde{a}_\rho \otimes_{\mathbb{C}} \tilde{t}_\eta$, as a local \mathfrak{g} -system on \tilde{E} , is trivial. Hence a flat section of $\tilde{a}_\rho \otimes_{\mathbb{C}} \tilde{t}_\eta$ is canonically identified with an element $x \in \mathfrak{g}$. On the other hand, a flat section of the local \mathfrak{g} -system $a_\rho \otimes t_\eta$ on E is a π_1 -invariant flat section of $\tilde{a}_\rho \otimes_{\mathbb{C}} \tilde{t}_\eta$. Therefore, a flat section of $a_\rho \otimes t_\eta^{-1}$ is canonically identified with an element $x \in \mathfrak{g}$ with the property that $q_i^{-1} \text{ad}(s_i)x = x$ for $i = 1, 2$. Recall that the $T^{\text{alg}} \times \mathbb{G}_m$ -action on \mathfrak{g} is by $(t, q) \cdot x = q^{-1} \text{ad}(t)x$ for $(t, q) \in T^{\text{alg}} \times \mathbb{G}_m$ and $x \in \mathfrak{g}$. Hence, $x \in \mathfrak{g}$ such that $q_i^{-1} \text{ad}(s_i)x = x$ for $i = 1, 2$ is the same as an element $x \in \mathfrak{g}^{T(a, t)}$.

(2). A B -structure on (a_ρ, x) is a Borel subgroup $B \subseteq G$ such that $s_i \in B$ for $i = 1, 2$, and $x \in \text{Lie}(B)$. Note that $s_i \in B$ if and only if $B \in \mathcal{B}^{s_i}$. Therefore, the B -structure on (a_ρ, x) are parametrized by $\mathcal{B}_x^{s_1, s_2}$.

(3). It is well-known that the group of automorphisms of a local system is the centralizer of the image of π_1 in G . Hence, $\text{Aut}(\rho_a) \cong G(a)$. This in term implies that the group of automorphisms of the pair (ρ_a, x) is $G(a, x)$. \square

Proof of Theorem 6.12. It is well-known that a holomorphic G -bundle on E admits a unitary flat connection if and only if it is semi-stable of degree zero (see e.g., [NS65]). For the adjoint bundle of such a G -bundle, a section is flat if and only if it is holomorphic. Therefore, Theorem 6.12 follows directly from Proposition 6.16. \square

7. REPRESENTATIONS AT TORSION POINTS IN TYPE-A

In this section we still work under the assumption that E is an elliptic curve over \mathbb{C} . We study the combinatorics related to representations of the elliptic affine Hecke algebra corresponding to U_n , when the parameter γ is evaluated at a torsion point $t \in E$.

7.1. Reminder on quiver Hecke algebras. Let Γ be an arbitrary finite symmetric quiver, with the set of vertices denoted by I , and the corresponding Cartan matrix denoted by C . For any pair of vertices $i, j \in I$, define the polynomial in two variables $P_{i,j}(u, v)$ to be $(v^{h/j \cdot j} - u^{h/i \cdot i})^{d_{i,j}}$ where $h = \text{lcm}(i \cdot i, j \cdot j)$ for $i \neq j$, and $P_{i,i}(u, v) = 0$.

Associated to Γ , there is a quiver Hecke algebra (also known as the KLR-algebra) $H_n(\Gamma)$ for any $n \geq 0$. See [R08, §3.2.1] for a presentation of this algebra, which we will not use in this paper. Instead, we recall the following fact (proved in [R08, Proposition 3.12] and [KL09, § 2.3]), which can be taken as the definition.

Let $\mathcal{O}' = \bigoplus_{\nu \in I^n} \mathbb{C}[x_1, \dots, x_n][\{(x_i - x_j)^{-1}\}_{i \neq j, \nu_i = \nu_j}]$. The idempotent in \mathcal{O}' corresponding to the direct summand labelled by $\nu \in I^n$ is denoted by 1_ν . Let $A_n(I) = \mathbb{C}^{(I)}[x] \wr \mathfrak{S}_n$, the wreath product. The variable x in the i -th $\mathbb{C}^{(I)}[x]$ -tensor factor of $A_n(I)$ will be denoted by x_i . Define $\tau_i \in \mathcal{O}' \otimes_{\mathbb{Z}^{(I)}[x]^{\otimes n}} A_n(I)$ as follows:

$$\tau_{i, \nu} = \begin{cases} \frac{s_i - 1}{x_i - x_{i+1}} 1_\nu, & \text{if } \nu_i = \nu_{i+1}, \\ P_{\nu_i, \nu_{i+1}}(x_{i+1}, x_i) s_i 1_\nu, & \text{otherwise.} \end{cases}$$

Proposition 7.1. *The algebra $H_n(\Gamma)$ is the subalgebra of $\mathcal{O}' \otimes_{\mathbb{Z}^{(I)}[x]^{\otimes n}} A_n(I)$ generated by 1_ν for $\nu \in I^n$, x_i for $i = 1, \dots, n$, and τ_i for $i = 1, \dots, n$.*

In particular, the algebra $H_n(\Gamma)$ admits a faithful representation on $\bigoplus_{\nu \in I^n} \mathbb{C}[x_1, \dots, x_n] 1_\nu$.

We recall some well-known facts about representations of the quiver Hecke algebras. The idempotents 1_ν define a direct sum decomposition

$$H_n(\Gamma) = \bigoplus_{\nu \in I^n} H_\nu(\Gamma).$$

The algebra $H_\nu(\Gamma)$ has a natural grading (see [KL09, § 2.1]), which is compatible with the decomposition $H_n(\Gamma) = \bigoplus_{\nu \in I^n} H_\nu(\Gamma)$. Let $\text{Mod}_0^{\text{gr}} H_n(\Gamma)$ be the abelian category of finite dimensional graded $H_n(\Gamma)$ -modules, and let $\text{Proj}^{\text{gr}} H_n(\Gamma)$ be the exact category of graded projective modules. Let $K^0(\text{Mod}_0^{\text{gr}} H_n(\Gamma))$ (resp. $K^0(\text{Proj}^{\text{gr}} H_n(\Gamma))$) be the Grothendieck group of $\text{Mod}_0^{\text{gr}} H_n(\Gamma)$ (resp. $\text{Proj}^{\text{gr}} H_n(\Gamma)$). They are $\mathbb{Z}[q^\pm]$ -algebras where q acts by degree shifting. Note that the Euler pairing induces an isomorphism $K^0(\text{Proj}^{\text{gr}} H_n(\Gamma))_{\mathbb{C}} \cong K^0(\text{Mod}_0^{\text{gr}} H_n(\Gamma))_{\mathbb{C}}^*$. Under this pairing, the basis on the left hand side formed by the classes of indecomposable projective objects are mapped to the dual basis of the classes of simple objects on the right hand side.

Observe that the natural embeddings $H_k(\Gamma) \otimes H_l(\Gamma) \hookrightarrow H_{k+l}(\Gamma)$ for any $l, k \in \mathbb{N}$ define induction functors

$$\text{Ind}_{k,l} : \text{Proj}^{\text{gr}} H_k(\Gamma) \otimes \text{Proj}^{\text{gr}} H_l(\Gamma) \rightarrow \text{Proj}^{\text{gr}} H_{k+l}(\Gamma),$$

and hence induce a multiplication on $\bigoplus_n K^0(\text{Proj}^{\text{gr}} H_n(\Gamma))$, making it an associative algebra.

We summarize the basic theory of quiver Hecke algebras as the following.

Theorem 7.2. *With notations as above, we have the following.*

- (1) *There is an isomorphism of $\mathbb{C}[q^\pm]$ -algebras*

$$\bigoplus_n K^0(\text{Proj}^{\text{gr}} H_n(\Gamma)) \cong U_{q^\pm}(\mathfrak{n}^-),$$

where \mathfrak{n}^- is the negative half of the Kac-Moody Lie algebra associated to the quiver Γ .

- (2) *For any $i \in I$, let $P(i) = H_1(\Gamma) \cdot 1_i$. Then the isomorphism above sends $[P(i)]$ to $f_i \in U_{q^\pm}(\mathfrak{n}^-)$.*
- (3) *The basis of the left hand side formed by the classes of indecomposable projective objects are mapped to the Lusztig canonical basis of the right hand side.*

The first two statements in Theorem 7.2 are proved in [KL09] and [R08]. The third statement is conjectured by Khovanov-Lauda and proved by Varagnolo-Vasserot in [VV11].

We can also state a version of Theorem 7.2 without grading. Let $\text{Mod}_0 H_n(\Gamma)$ be the category of finite dimensional $H_n(\Gamma)$ -modules on which x_i acts nilpotently for any $i = 1, \dots, n$. The inclusion

$$H_{n-1}(\Gamma) \otimes H_1(\Gamma) \hookrightarrow H_n(\Gamma)$$

induces a restriction of scalars $\text{Mod}_0 H_n(\Gamma) \rightarrow \text{Mod}_0(H_{n-1}(\Gamma) \otimes H_1(\Gamma))$. For any $i \in I$, the right multiplication by the idempotent $1_i \in H_1(\Gamma)$ defines $\text{Mod}_0(H_{n-1}(\Gamma) \otimes H_1(\Gamma)) \rightarrow \text{Mod}_0 H_{n-1}(\Gamma)$. Let

$$\text{Res}_i : \text{Mod}_0 H_n(\Gamma) \rightarrow \text{Mod}_0 H_{n-1}(\Gamma)$$

be their composition. Then there is a \mathbb{C} -algebra isomorphism

$$\bigoplus_n K^0(\text{Mod}_0 H_n(\Gamma))_{\mathbb{C}}^* \cong U(\mathfrak{n}^-).$$

This isomorphism intertwines the operation $[\text{Res}_i]^*$ on the left hand side and multiplication by f_i on the right hand side.

7.2. Completion of \mathcal{H}_n and the quiver Hecke algebra. From now on \mathcal{H}_n will be the elliptic affine Hecke algebra of U_n . We will drop the lower subscript n if it is understood from the context. We have $\mathfrak{A}_G = E^{(n)}$, $\mathfrak{A}/W = E^{(n)} \times E$, and $\mathcal{S} = \pi_* \mathcal{O}_{E^{n+1}}$ where $\pi : E^{n+1} \rightarrow E^{(n)} \times E$ is the symmetrization map. For any $\nu \in \mathfrak{A} = E^{n+1}$, we write it in coordinates as $\nu = (\nu_1, \dots, \nu_n, \nu_\gamma)$. The following lemma is a direct consequence of the structure theorem. Let \mathfrak{l} be an arbitrary local parameter of the elliptic curve E .

Lemma 7.3. *Recall that the divisor D^i is given by the equation $\nu_i = \nu_{i+1}$, and $D^{i,\gamma}$ is given by $\nu_{i+1} - \nu_i = \nu_\gamma$.*

- (1) *Let $\tilde{\mathfrak{A}}^c = \mathfrak{A} \setminus (\cup_{1 \leq i \leq n-1} (D^i \cup D^{i,\gamma}))$. The sheaf of algebras $\mathcal{H}|_{\tilde{\mathfrak{A}}^c/W}$ is isomorphic to $\mathcal{S}_W|_{\tilde{\mathfrak{A}}^c/W}$.*
- (2) *Let $\nu \in E^{n+1}$ be a closed point. Let \mathcal{O}_ν^\wedge be the completion of $\mathcal{O}_{E^{n+1}}$ at ν . Then, the algebra structure on \mathcal{H} induces an algebra structure on*

$$\mathcal{H}_{\mathfrak{S}_n \cdot \nu}^\wedge := \left(\bigoplus_{\mu \in \mathfrak{S}_n \cdot \nu} \pi_* \mathcal{O}_\mu^\wedge \right) \otimes_{\mathcal{S}} \mathcal{H}.$$

- (3) *The algebra $\mathcal{H}_{\mathfrak{S}_n \cdot \nu}^\wedge$ is a subalgebra of $\text{End}(\bigoplus_{\mu \in \mathfrak{S}_n \cdot \nu} \pi_* \mathcal{O}_\mu^\wedge)$, generated by $\bigoplus_{\mu \in \mathfrak{S}_n \cdot \nu} \pi_* \mathcal{O}_\mu^\wedge$ and the operators T_i with $i = 1, \dots, n-1$ such that*

$$(T_i)_\mu := \begin{cases} \frac{\mathfrak{l}(\gamma)}{\mathfrak{l}(x_{i+1} - \mu_{i+1}) - \mathfrak{l}(x_i - \mu_i)} (1_\mu - s_i) + s_i, & \text{if } \mu \in D^i; \\ (\mathfrak{l}(x_{i+1} - \mu_i) - \mathfrak{l}(x_i - \mu_i) - \mathfrak{l}(\gamma - \mu_\gamma)) s_i, & \text{if } \mu \in D^{i,\gamma}; \\ s_i, & \text{otherwise.} \end{cases}$$

- (4) *Evaluating $\gamma = \nu_\gamma$, we get that the algebra $(\mathcal{H}_{\mathfrak{S}_n \cdot \nu}^\wedge)_{\gamma=\nu_\gamma}$ is generated by*

$$\left(\bigoplus_{\mu \in \mathfrak{S}_n \cdot \nu} \pi_* \mathcal{O}_\mu^\wedge \right) / (\gamma = \nu_\gamma)$$

and the operators T_i :

$$(T_i)_\mu := \begin{cases} \frac{\mathfrak{l}(\nu_\gamma)}{\mathfrak{l}(x_{i+1}) - \mathfrak{l}(x_i)} (1_\mu - s_i) + s_i, & \text{if } \mu \in D^i; \\ (\mathfrak{l}(x_{i+1} - \mu_{i+1}) - \mathfrak{l}(x_i - \mu_i)) s_i, & \text{if } \mu \in D^{i,\gamma}; \\ s_i, & \text{otherwise.} \end{cases}$$

In (3) and (4), the primitive idempotent element in $\bigoplus_{\mu \in \mathfrak{S}_n \cdot \nu} \pi_* \mathcal{O}_\mu^\wedge$ corresponding to the multiplicative identity in $\pi_* \mathcal{O}_\mu^\wedge$ is denoted by 1_μ . The operators s_i should be understood as going from $\pi_* \mathcal{O}_\mu^\wedge$ to $\mathcal{O}_{s_i \mu}^\wedge$.

Let $t \in E$ be a torsion point. We naturally identify E^n with $E^n \times \{t\} \subseteq E^{n+1}$. Then $\mathcal{H}_n / (\nu_\gamma = t)$ is a sheaf of algebras on $E^{(n)}$. Suppose that $DD(t) = (q_1, q_2) \in (\mathbb{C}^*)^2$, where DD is the map defined in § 6.2. Suppose that q_1 is of order n_1 , and q_2 is of order n_2 , where n_1, n_2 are integers strictly greater than 1. Let $S_t \subset E$ be the subset consisting of $z \in E$ such that $DD(z)$ is of the form $(q_1^u, q_2^v) \in \mathbb{C}^* \times \mathbb{C}^*$ for $u, v \in \mathbb{Z}$. Let $S_t^n \subseteq E^n$ be the subset of E^n consisting of points whose coordinates are in S_t . Let $\text{Mod}_t \mathcal{H}_n$ be the subcategory of finite dimensional \mathcal{H}_n -modules, whose restriction to the action of \mathcal{S} , considered as a coherent sheaf on $E^{(n)}$, is set-theoretically supported on $\pi(S_t^n)$.

We have the following commutative diagram

$$\begin{array}{ccc} E^k \times E^l & \xlongequal{\quad} & E^{k+l} \\ \downarrow & & \downarrow \\ E^{(k)} \times E^{(l)} & \xrightarrow{\pi_{k,l}} & E^{(k+l)}. \end{array}$$

Consider $\mathcal{H}_k \boxtimes \mathcal{H}_l$ as a coherent sheaf of algebras on $E^{(k)} \times E^{(l)}$. By definition of the elliptic affine Hecke algebras, $\pi_{k,l*} \mathcal{H}_k \boxtimes \mathcal{H}_l$ is a subsheaf of algebras of $\mathcal{O} \text{nd}_{E^{(k+l)}}(\pi_* \mathcal{O}_{E^{k+l}})$. It is easy to verify that $\pi_{k,l*} \mathcal{H}_k \boxtimes \mathcal{H}_l$ lies in the subsheaf \mathcal{H}_{k+l} . Therefore, we have an injective map $\pi_{k,l*} \mathcal{H}_k \boxtimes \mathcal{H}_l \rightarrow \mathcal{H}_{k+l}$, which induces induction and restriction functors on the category of representations. One can easily check that when restricting to subcategories of representations, the following functors

$$\begin{array}{ccc} \text{Mod-}_t \mathcal{H}_k \otimes \text{Mod-}_t \mathcal{H}_l & \xrightarrow{\text{Ind}} & \text{Mod-}_t \mathcal{H}_{k+l} \\ & \xleftarrow{\text{Res}} & \end{array}$$

are well-defined

Let $d = \text{lcm}(n_1, n_2)$ and let $l = \frac{n_1 n_2}{d}$. Let $\Gamma = \Gamma_{d,l}$ be the disjoint union of l quivers, each of type $A_{d-1}^{(1)}$. Note that Γ naturally embeds into E as follows. Let I be S_t inside E . For $z, z' \in I$ with $DD(z) = (q_1^u, q_2^v)$ and $DD(z') = (q_1^{u+1}, q_2^{v+1})$, there is a single arrow from z to z' . One can simply verify that the quiver obtained this way is isomorphic to $\Gamma_{d,l}$. In particular, the set of vertices I can be relabelled by the set $\{(i, j) \mid i = 0, \dots, d-1, \text{ and } j = 1 \dots, l\}$.

Remark 7.4. It follows from § 3.6 and Theorem 5.6 that $(\mathcal{H}_n)_{S_t^n/\mathfrak{S}_n}^\wedge \cong \bigoplus_{\mu \in S_t^n/\mathfrak{S}_n} H_{G(\mu,t)}^*(Z^{T(\mu,t)})$. It is not hard to show that the right hand side is isomorphic to the quiver Hecke algebra for the quiver $\Gamma_{d,l}$. The induction and restriction functors can also be described geometrically similar to [AJL08, § 4]. However, thanks to Lemma 7.3, it turns out to be easier to follow the purely algebraic approach of Rouquier in [R08].

Let $\mathcal{O}^\wedge = \bigoplus_{\nu \in I^n} \mathbb{C}[x_1, \dots, x_n]_{x=\nu}^\wedge$, and let the idempotent corresponding to ν be denoted by 1_ν . Define an algebra structure on

$$\mathcal{O}^\wedge H_n(\Gamma) := \mathcal{O}^\wedge \otimes_{(\bigoplus_{\nu \in I^n} \mathbb{C}[x_1, \dots, x_n] 1_\nu)} H_n(\Gamma)$$

by setting

$$\tau_i 1_\nu - 1_{s_i \nu} \tau_i = (x_{i+1} - x_i)^{-1} (1_\nu - 1_{s_i \nu}).$$

Here s_i is the simple reflection exchanging i and $i+1$. Let the operator T_i be defined as in Lemma 7.3.

Lemma 7.5. *The following assignment induces an isomorphism of algebras*

$$\phi : \mathcal{O}^\wedge H_n(\Gamma_{d,l}) \rightarrow \bigoplus_{\mu \in S_t^n/\mathfrak{S}_n} (\mathcal{H}_n)_\mu^\wedge$$

where

$$\begin{aligned} \phi(x_i 1_\nu) &= \mathfrak{l}(z_i - \nu_i) 1_\nu \\ \phi(\tau_i) &= \frac{T_i - 1}{\mathfrak{l}(z_{i+1} - \nu_{i+1}) - \mathfrak{l}(z_i - \nu_i) - \mathfrak{l}(t)} 1_\nu = \frac{s_i - 1}{\mathfrak{l}(z_{i+1} - \nu_{i+1}) - \mathfrak{l}(z_i - \nu_i)} 1_\nu & \text{if } \nu_{i+1} = \nu_i \\ \phi(\tau_i) &= T_i 1_\nu = (\mathfrak{l}(z_{i+1} - \nu_{i+1}) - \mathfrak{l}(z_i - \nu_i)) s_i 1_\nu & \text{if } \nu_{i+1} = \nu_i + t \\ \phi(\tau_i) &= (\mathfrak{l}(z_{i+1} - \nu_{i+1}) - \mathfrak{l}(z_i - \nu_i)) s_i 1_\nu & \text{otherwise.} \end{aligned}$$

Proof. This Lemma followed directly from [R08, Proposition 3.12] (which is recalled above as Proposition 7.1) and Lemma 7.3. \square

Due to this Lemma, there is a \mathbb{G}_m -action on $\bigoplus_{\mu \in S_t^n / \mathfrak{S}_n} (\mathcal{H}_n)_\mu^\wedge$ coming from the \mathbb{G}_m -action on $\mathcal{O}^\wedge H_n(\Gamma_{d,l})$. Hence, there is a well-defined notion of finite dimensional graded modules over \mathcal{H}_n supported on S_t^n . We denote the category of such modules by $\text{Mod}_t^{\text{gr}} \mathcal{H}_n$. For each pair of integers $k, l \in \mathbb{N}$, the induction and restriction functors

$$\begin{array}{ccc} \text{Mod}_t^{\text{gr}} \mathcal{H}_k \otimes \text{Mod}_t^{\text{gr}} \mathcal{H}_l & \xrightarrow{\text{Ind}} & \text{Mod}_t^{\text{gr}} \mathcal{H}_{k+l} \\ & \xleftarrow{\text{Res}} & \end{array}$$

are well-defined, and are compatible with the forgetful functors $\text{Mod}_t^{\text{gr}} \mathcal{H}_n \rightarrow \text{Mod}_t \mathcal{H}_n$.

Summarizing the discussion above, we have the following.

Theorem 7.6. *With notations as above, we have*

- (1) *There is an equivalence of abelian categories $\text{Mod}_0^{\text{gr}} H_n(\Gamma_{d,l}) \cong \text{Mod}_t^{\text{gr}} \mathcal{H}_n$ which is compatible with Ind and Res on both sides.*
- (2) *There is an equivalence of abelian categories $\text{Mod}_0 H_n(\Gamma_{d,l}) \cong \text{Mod}_t \mathcal{H}_n$ which is compatible with Ind and Res on both sides.*
- (3) *The equivalences in (1) and (2) intertwines with the forgetful functors $\text{Mod}_t^{\text{gr}} \mathcal{H}_n \rightarrow \text{Mod}_t \mathcal{H}_n$ and $\text{Mod}_0^{\text{gr}} H_n(\Gamma_{d,l}) \rightarrow \text{Mod}_0 H_n(\Gamma_{d,l})$.*

For each $(i, j) \in I$, we define functors $\text{Res}_{(i,j)} : \text{Mod}_t^{\text{gr}} \mathcal{H}_n \rightarrow \text{Mod}_t^{\text{gr}} \mathcal{H}_{n-1}$ as follows. For any module $M \in \text{Mod}_t^{\text{gr}} \mathcal{H}_n$, by definition, M decomposes into $\bigoplus_{(u,v) \in S_t} M_{(u,v)}$ as a coherent sheaf on E^n . Here for each $(u, v) \in S_t$, the submodule $M_{(u,v)}$ is the direct summand of M as coherent sheaf on E^n , whose support has the n -th coordinate equal to $(u, v) \in S_t \subset E$. Consider M as a module over \mathcal{H}_{n-1} via the map $\mathcal{H}_{n-1} \boxtimes 1 \subseteq \mathcal{H}_{n-1} \boxtimes \mathcal{H}_1 \rightarrow \mathcal{H}_n$. Obviously, each $M_{(u,v)}$ is a \mathcal{H}_{n-1} -submodule of M . We define $\text{Res}_{(i,j)}(M)$ to be the direct summand of M as \mathcal{H}_{n-1} -module whose support has the n -th coordinate equal to $(i, j) \in I \subseteq E$.

Similarly to the graded situation, the functors $\text{Res}_{(i,j)} : \text{Mod}_t \mathcal{H}_n \rightarrow \text{Mod}_t \mathcal{H}_{n-1}$ are also well-defined.

Let $U_q(\widehat{\mathfrak{sl}}_d)$ be the affine quantum group of $\widehat{\mathfrak{sl}}_d$, and $U_q^-(\widehat{\mathfrak{sl}}_d)$ its negative part. Let

$$f_{ij} = 1 \otimes \cdots \otimes 1 \otimes f_j \otimes 1 \otimes \cdots \otimes 1 \in U_q^-(\widehat{\mathfrak{sl}}_d)^{\otimes l}$$

where f_j is in the i -th tensor factor.

- Corollary 7.7.** (1) *There is an \mathbb{C} -linear isomorphism $U_q^-(\widehat{\mathfrak{sl}}_d)^{\otimes l} \rightarrow \bigoplus_n K(\text{Mod}_t^{\text{gr}} \mathcal{H}_n)^*$.*
 (2) *This isomorphism intertwines multiplication of f_{ij} on the left and $[\text{Res}_{i,j}]^*$ on the right.*
 (3) *Under this isomorphism, the basis on $K(\text{Mod}_t^{\text{gr}} \mathcal{H}_n)^*$ dual to the classes of simple objects corresponds to the Lusztig canonical basis on $U_q^-(\widehat{\mathfrak{sl}}_d)^{\otimes l}$.*
 (4) *This isomorphism induces an isomorphism $U^-(\widehat{\mathfrak{sl}}_d)^{\otimes l} \cong \bigoplus_n K(\text{Mod}_t \mathcal{H}_n)^*$, which also intertwines $[\text{Res}_{i,j}]^*$ and f_{ij} .*

This is a direct corollary of Theorem 7.2 and Theorem 7.6.

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